STEADY STATE ACCURACY

Consider the unity feedback system

\[ R(s) \xrightarrow{+} G_c(s) \xrightarrow{G_p(s)} C(s) \]

The \textbf{SYSTEM TYPE NUMBER} \( N \) is the number of pure integrators in the forward path \( G_c(s) G_p(s) \), i.e., the number of poles at \( s = 0 \) of \( G_c(s) G_p(s) \).

\begin{equation}
G_c(s) G_p(s) = \frac{G(s)}{s^N p(s)}
\end{equation}

is \textbf{TYPE N} as long as \( q(0) \) and \( p(0) \) are not equal to 0. It doesn't matter if the factors of \( s \) come from the plant \( G_p(s) \) or from the compensator \( G_c(s) \).
Examples:

**Type 0 - System**

\[ R \xrightarrow{} K_p \xrightarrow{} \frac{K}{\tau s + 1} \rightarrow C \]

**Type 1 - System**

\[ R \xrightarrow{} \frac{K_p + K_i}{s} \xrightarrow{} \frac{K}{\tau s + 1} \rightarrow C \]

\[ G_c(s) G_p(s) = \frac{(K_p s + K_i) K}{s (\tau s + 1)} \]

**Type 2 - System**

\[ R \xrightarrow{} K_p \xrightarrow{} \frac{K}{s(\tau s + 1)} \rightarrow C \]
The system type number is important because it determines the steady state accuracy of the system. We assume that the closed loop system is stable. Otherwise there is no steady state. We consider the following types of reference inputs:

1) **step input**
   \[ r(t) = A u(t) \quad ; \quad R(s) = \frac{A}{s} \]

2) **ramp input**
   \[ r(t) = A t u(t) \quad ; \quad R(s) = \frac{A}{s^2} \]

3) **parabola input**
   \[ r(t) = A \frac{t^2}{2} u(t) \quad ; \quad R(s) = \frac{A}{s^3} \]
we compute the error transfer function $E(s)/R(s)$ as follows:

$$E = R - C = R - \frac{G_c G_p}{1 + G_c G_p} R$$

$$= \frac{1}{1 + G_c G_p} R$$

Now, the steady state error is computed using the Final Value Theorem as:

$$e_{ss} = \lim_{s \to 0} s \cdot \frac{1}{1 + G_c(s) G_p(s)} R(s)$$

$$= \lim_{s \to 0} s \cdot \frac{1}{1 + \frac{G(s)}{s^n p(s)}} R(s)$$

$$= \lim_{s \to 0} \frac{s^{n+1} p(s)}{s^n p(s) + q(s)} R(s)$$
Examples: If \( R(s) = \frac{A}{s} \) (step input)

Then

\[
\mathcal{E}_s = \lim_{s \to 0} \frac{s^{N+1} \rho(s)}{s^N \rho(s) + \varrho(s)} = A \frac{s^N \rho(s)}{s^N \rho(s) + \varrho(s)}
\]

\[
= A \frac{\rho(0)}{\rho(0) + \varrho(0)} \quad \text{if} \quad N = 0
\]

\[
= 0 \quad \text{if} \quad N \geq 1
\]

If \( R(s) = \frac{A}{s^2} \) (ramp input)

Then

\[
\mathcal{E}_s = \lim_{s \to 0} A \frac{s^{N-1} \rho(s)}{s^N \rho(s) + \varrho(s)}
\]

\[
= \infty \quad \text{if} \quad N = 0
\]

\[
= A \frac{\rho(0)}{\varrho(0)} \quad \text{if} \quad N = 1
\]

\[
= 0 \quad \text{if} \quad N \geq 2
\]
We define the error constant $\kappa_p, \kappa_r, \kappa_a$ (Kappa) as:

$$\kappa_p = \lim_{s \to 0} G_c(s) \cdot G_p(s)$$

$$\kappa_r = \lim_{s \to 0} s \cdot G_c(s) \cdot G_p(s)$$

$$\kappa_a = \lim_{s \to 0} s^2 \cdot G_c(s) \cdot G_p(s)$$

We can see from this that

$$\kappa_p = G_c(0) \cdot G_p(0) < \infty \quad \text{if } N = 0$$

and

$$\kappa_p = \infty \quad \text{for } N \geq 1$$

$$\kappa_r = 0 \quad \text{for } N = 0$$

$$= G_c(0) \cdot G_p(0) < \infty \quad \text{for } N = 1$$

$$= \infty \quad \text{for } N \geq 2$$

$$\kappa_a = 0 \quad \text{for } N = 0, 1$$

$$= G_c(0) \cdot G_p(0) < \infty \quad \text{for } N = 2$$

$$= \infty \quad \text{for } N \geq 3$$
we can use this to develop the following table which gives the steady state errors

<table>
<thead>
<tr>
<th>R</th>
<th>1/s</th>
<th>1/s^2</th>
<th>1/s^3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(\frac{1}{1+K_p})</td>
<td>(\infty)</td>
<td>(\infty)</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>(\frac{1}{X_u})</td>
<td>8</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>(\frac{1}{X_a})</td>
</tr>
</tbody>
</table>

Example:

TYPE 0 SYSTEM  
\[K_p = G_e(0)G_p(0) = K K_p\]

Thus if \(R = \frac{A}{s}\)  
\[\varepsilon_{ss} = A \cdot \frac{1}{s(1+KKp)}\]
and \(\varepsilon_{ss} \to 0\) as \(K_p \to \infty\)
Type 1 - system, we therefore know that, if \( R(s) = \frac{A}{s} \), the \( e_{ss} = 0 \) for any values of the parameters as long as the system is stable.

**Summary:** To "track" a step input the system must be at least type 2. To track a ramp input the system must be at least type 2, etc.
applications

\[ R(s) \xrightarrow{+} Q \xrightarrow{K} \frac{2}{s^3 + 4s^2 + 32} \xrightarrow{C(s)} \]

- For what values of \( K \) is the closed loop system stable?

- How small can the steady state error to a unit step be made?

Solution: The closed loop characteristic equation is
\[ 1 + G_c G_p = 1 + \frac{2K}{s^3 + 4s^2 + 5s + 2} = 0 \]

or

\[ Q(s) = \frac{s^3 + 4s^2 + 5s + 2 + 2K}{s^3 + 4s^2 + 5s + 2} = 0 \]

1) since \( 2 + 2K \) must be positive we must have \( K > -1 \)

2) since \( a_2 a_1 > a_0 \) we must have

\[ 4.5 > 2 + 2K \]

\[ \Rightarrow K < 9 \]

Therefore the system is stable only for

\[ -1 < K < 9 \]
The steady state error to a unit step is

\[ e_{ss} = \frac{1}{1 + K_p} \]

where \( K_p = \lim_{s \to 0} G_c(s) G_p(s) \)

\[ = \lim_{s \to 0} \frac{2K}{s^3 + 4s^2 + 5s + 2} = K \]

Therefore

\[ e_{ss} = \frac{1}{1 + K} \]

But only if the system is stable. Since \( K < 9 \) the smallest steady state error possible is

\[ e_{ss} = \frac{1}{10} \]
$K = 2$

$K = 5$
We know that we can obtain a zero steady state error by adding an integral control to make the system Type 1.

\[ \text{PI} \]

For what values of the gains \( K_p \) and \( K_i \) is the closed loop system stable? The system characteristic equation is

\[
1 + \frac{2}{s(s^2 + 4s^2 + 5s + 2)} = 0
\]
The characteristic polynomial is

\[ Q(s) = s^4 + 4s^3 + 5s^2 + (2 + 2K_p)s + 2K_i \]

The Routh-Hurwitz array is

<table>
<thead>
<tr>
<th>$s^4$</th>
<th>1</th>
<th>5</th>
<th>$2K_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s^3$</td>
<td>$4$</td>
<td>$2 + 2K_p$</td>
<td></td>
</tr>
<tr>
<td>$s^2$</td>
<td>$\frac{18 - 2K_p}{4}$</td>
<td>$2K_i$</td>
<td>$K_0 &lt; 9$</td>
</tr>
<tr>
<td>$s^1$</td>
<td>$C_1$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$s^0$</td>
<td>$2K_i$</td>
<td></td>
<td>$K_2 &gt; 0$</td>
</tr>
</tbody>
</table>

where

\[ C_1 = \frac{-4}{18 - 2K_p} \left( 8K_i - \frac{(2 + 2K_p)(18 - 2K_p)}{4} \right) \]

\[ = \frac{4}{18 - 2K_p} \left[ (1 + K_p)(9 - K_p) - 8K_i \right] \]
we see that we must have
\[ K_P < 9 \]
\[ K_I > 0 \]
\[ C_i > 0 \]

There are infinitely many choices for \( K_P, K_I \) that result in a stable system with zero steady state error. Other specifications would be used to narrow down the choices.

Try \( K_P = 2 \) \( \checkmark \) Then
\[ C_i = \frac{4}{16} [2.8 - 8K_I] \Rightarrow K_I < 2 \]

The response for \( K_I = 1 \) is shown next.
$K_p = 1$

$K_i = 1$
As $K_p, K_2$ get closer to the limit for stability, the response becomes more oscillatory. For example, with $K_p = 5$,

$$C_1 = 2 \left[ 24 - 8K_2 \right] \Rightarrow K_2 < 3$$

The response with $K_p = 5$, $K_2 = 2$ is shown below:

![Graph showing oscillatory response with time and amplitude axes.]