Chapter 5: Stability Analysis

Recall that a linear time-invariant (LTI) system is **BIBO stable** if all of its poles are in the open left half of the $s$-plane and **marginally stable** if all poles are in the left half plane except for simple poles on the $jw$-axis, and **unstable** if any poles are in the right half plane.
The question of stability reduces to finding the location of the roots of the characteristic polynomial. Note that we don't need the exact values of the roots, only the sign of the real part to determine stability. This leads to the Routh-Hurwitz criterion.
Given a (characteristic) polynomial $Q(s) = s^n + a_{n-1}s^{n-1} + \ldots + a_1s + a_0$

we would like to test using only the $n$-coefficients $a_0, a_1, \ldots, a_{n-1}$ whether or not all roots of $Q(s)$ are in the left half plane.

[Such a polynomial is called a Hurwitz polynomial]
Some examples:

2nd order:

\[ Q(s) = s^2 + a_1 s + a_0 \]

Suppose that both roots of \( Q(s) \) are real and in the left half plane.

\[ Q(s) = (s - p_1)(s - p_2) \]

with \( p_1, p_2 < 0 \). Then

\[ Q(s) = s^2 - (p_1 + p_2)s + p_1 p_2 \]

\[ \Rightarrow a_1 = -(p_1 + p_2) \]

\[ a_0 = p_1 p_2 \]
\( Q(s) = s^3 + a_2 s^2 + a_1 s + a_0 \)

\[ = (s - p_1)(s - p_2)(s - p_3) \]

\[ = s^3 - (p_1 + p_2 + p_3) s^2 + (p_1 p_2 + p_1 p_3 + p_2 p_3) s - p_1 p_2 p_3 \]

\[ \Rightarrow \quad a_2 = -(p_1 + p_2 + p_3) \]

\[ a_1 = p_1 p_2 + p_1 p_3 + p_2 p_3 \]

\[ a_0 = -p_1 p_2 p_3 \]

What is the pattern?
\[ s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \cdots + a_1 s + a_0 \]
\[ = (s - p_1)(s - p_2) \cdots (s - p_n) \]
\[ \Rightarrow a_{n-1} = -\sum_{i=1}^{n} p_i \]
\[ = -\text{(sum of all roots)} \]

\( a_{n-2} = \text{sum of products of roots taken 2 at a time} \)

\( a_{n-3} = -\text{(sum of products of roots taken 3 at a time)} \)

\[ a_0 = (-1)^n \cdot \text{product of all roots} \]
we see that if all the $\pi_i$ are negative then

\[ a_{n-1} = -(\text{sum of roots}) > 0 \]

\[ a_{n-2} = \text{sum of products of 2 roots} > 0 \]

\[ \vdots \]

\[ a_0 = (-1)^n \cdot \text{product of roots} > 0 \]

Thus, all of the coefficients must be positive!!

This gives us an easy test for stability

(Actually instability)
Examples:

\[ s^3 - 3s^2 + 2s + 1 = Q(s) \]
\[(\text{unstable})\]

\[ s^4 + 3s^3 + s = Q(s) \]
\[(\text{unstable})\]

However, be careful

\[ Q(s) = s^3 + s^2 + 2s + 8 \]
\[ = \text{unstable because} \]

\[ Q(s) = (s+2)(s^2 - s + 4) \]
We can say more:

Notice:

\[(5+1)(5+2) = 5^2 + 35 + 2\]
\[(5-1)(5+2) = 5^2 + 5 - 2\]
\[(5+1)(5-2) = 5^2 - 5 - 2\]
\[(5-1)(5-2) = 5^2 - 35 + 2\]

Count the "sign changes"

\[+5^2 + 35 + 2 = 0 \text{ changes}\]
\[+5^2 + 5 - 2 = 2 \text{ sign changes}\]
\[+5^2 - 5 - 2 = 2 \text{ sign changes}\]
\[+5^2 - 35 + 2 = 2 \text{ sign changes}\]

\# of sign changes =

\# of unstable roots !!
However, this doesn't always work as the example $s^3 + s^2 + 2s + 5$
has already shown us. The Routh-Hurwitz test provides the correct method for "Counting sign changes" that always works.
Construction of the Routh-Hurwitz Array

Given the polynomial

\[ Q(s) = a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0 \]

we create a table of numbers as follows:

\[ \begin{array}{c|cccc}
   s^n & a_n & a_{n-2} & a_{n-4} & \ldots \\
   s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \ldots \\
   s^{n-2} & b_1 & b_2 & b_3 & \ldots \\
   \vdots & c_1 & c_2 & c_3 & \ldots \\
   s^1 & \vdots & \vdots & \vdots & \vdots \\
   s^0 & \vdots & \vdots & \vdots & \vdots \\
\end{array} \]
The elements $c_1, b_2, \ldots, c_1, c_2, \ldots, \text{ etc.}$ are computed as follows:

$$b_1 = -\frac{1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix}$$

$$= -\frac{1}{a_{n-1}} \left( a_n a_{n-3} - a_{n-1} a_{n-2} \right)$$

$$b_2 = -\frac{1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix}$$

$$c_1 = -\frac{1}{b_1} \begin{vmatrix} a_{n-1} & a_{n-3} \\ b_1 & b_2 \end{vmatrix}$$

\text{ etc.}
Example:

\[ Q(s) = s^3 + s^2 + 2s + 8 \]

\[
\begin{array}{c|ccc}
  \text{s}^3 & 1 & 2 \\
  \text{s}^2 & 1 & 8 \\
  \text{s} & -6 & 0 \\
  \text{r} & 8 \\
\end{array}
\]

- You only need to compute as many entries as needed to complete the first column.

- You can fill rows with zero terms if needed as in the above.
Routh–Hurwitz Criterion

The number of roots of \( Q(s) \) in the right half plane is equal to the number of sign changes in the first column of the Routh–Hurwitz array for \( Q(s) \).
Example:

\[ Q(s) = s^3 + s^2 + 2s + 8 \]

\[
\begin{array}{c|ccc}
  & s^3 & 1 & 2 \\
  & s^2 & 1 & 8 \\
  & s^1 & -6 \\
  & s^0 & 8 \\
\end{array}
\]

Therefore, \( Q(s) \) has 2 roots with positive real part since there are 2 sign changes in the first column of the Routh-Hurwitz array.
Corollary:

The polynomial $Q(s)$ is Hurwitz (i.e. stable) if and only if there are no sign changes in the first column of the Routh–Hurwitz array.