More on Phase Margin

With \( G(s) = \frac{\omega_n^2}{s(s+2\omega_n)} \): \( T(s) = \frac{G(s)}{1+G(s)} = \frac{\omega_n^2}{s^2+2s\omega_n+s\omega_n^2} \) the relation between \( PM \) and \( S \) is

\[
PM = \tan^{-1} \left( \frac{2s}{\sqrt{1+4s^2-2\zeta^2}} \right) \approx 100\zeta
\]

Bode’s Gain-Phase Relationship

[For any stable minimum phase system, the phase of \( G(j\omega) \) is uniquely related to the magnitude of \( G(j\omega) \).]

\[
\angle G(j\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} \left( \frac{dM}{du} \right) W(u) du \text{ is radians}
\]
where

\[
M = \ln|G(j\omega)| = \log \text{magnitude}
\]
\[
u = \ln(\omega/\omega_0) = \text{normalized frequency}
\]
\[
\omega(u) = \ln(\coth |u|/2)
\]

When the slope of \(|G(j\omega)| \) versus \( \omega \) on a log-log scale is constant over approximately a decade then \( \angle G(j\omega) \approx n \cdot 90^\circ \) which is used in practice more often when \(|KG(j\omega)| = 1 \) we have

\[
\angle G(j\omega) \approx -90^\circ \text{ if } n = -1(-20\text{db/dec})
\]
\[
\angle G(j\omega) \approx -180^\circ \text{ if } n = -2(-40\text{db/dec})
\]

Example:

**Design rule of thumb**  For stability we want \( \angle G(j\omega) > -180^\circ \) for a \( PM > 0 \). Therefore we adjust the \( |KG(j\omega)| \) curve so that it has a slope of \(-1\) at the crossover frequency \( \omega_c \). If the slope is \(-1\) for a decade above and below the crossover frequency, the \( PM \approx 90^\circ \).

Most often it is good enough to halve that, i.e., the slope should be \(-1\) for a frequency decade *centered* at \( \omega_0 \).

**Example:**

Choose \( KD(s) \) for good damping and a bandwidth of \( \approx 0.2 \text{ rad/sec.} \)
The open-loop Bode magnitude plot is shown. Choose a PD-compensator \( KD(s) = K(T_D s + 1) \). We choose the breakpoint \( \omega_1 = 1/T_D \) to provide a slope of \(-1\) at the crossover frequency and adjust \( K \) for the desired bandwidth.

The Bode plot for \( \frac{K(T_D s + 1)}{s^2} \) has the shape shown below:

**Step 1:** Plot \(|G(j\omega)|\)

**Step 2:** Pick \( \omega_1 = 0.2/4 = 0.05(T_D = 20) \)

**Step 3:** Determine that \(|DG| = 100\)

\[
|DG|_{s=0.2j} = \frac{|20s + 1|}{s^2} \bigg|_{s=0.2j} = |\frac{4j + 1}{0.04}| = 100
\]

**Step 4:** We want \( K|DG| = 1 \) implies \( K = \frac{1}{100} \). Therefore \( KD(s) = 0.01(20s + 1) \).

The closed-loop frequency response and step response are shown below:

Note in this example, the bandwidth was identical to the crossover frequency, i.e., \( \omega_{BW} = \omega_c = 0.2 \). In general, for a second-order system \( \omega_c \leq \omega_{BW} \leq 2\omega_c \). The actual value depends on the damping ratio and hence the phase margin.

**Compensation using Bode Plots**
Given $G(s)$, $KD(s)G(s)$ has desired characteristics. We will loop at $PD$, Lead, PI, and Lag Compensators.

Frequency Response for a $PD$-compensator

$$KD(s) = K(T_Ds + 1)$$

Frequency Response for a Lead Compensator

$$KD(s) = \frac{T_s + 1}{\alpha Ts + 1}, \alpha < 1, \quad \alpha = 0.1$$

$\alpha$ is the ratio between the pole/zero breakpoint frequencies. The phase contribution is

$$Q = \tan^{-1}(T\omega) - \tan^{-1}(\alpha T\omega).$$

The maximum phase occurs at

$$\omega_{\text{max}} = \frac{1}{T\sqrt{\alpha}}$$

and is given by

$$\sin Q_{\text{max}} = \frac{1 - \alpha}{1 + \alpha}$$

or

$$\alpha = \frac{1 - \sin d_{\text{max}}}{1 + \sin d_{\text{max}}}$$

another way to look at this

$$\log \omega_{\text{max}} = \log \frac{1}{\sqrt{\alpha T}} = \log \frac{1}{\sqrt{T}} + \log \frac{1}{\sqrt{\alpha T}}$$

$$= \frac{1}{2} \left( \log \frac{1}{T} + \log \frac{1}{\alpha T} \right)$$
with

\[ D(s) = \frac{s + z}{s + p} \]

\[ \omega_{\text{max}} = \sqrt{|z| \cdot |p|} \]

and

\[ \log \omega_{\text{max}} = \frac{1}{2} (\log |z| + \log |p|) \]

For example, if \( z = -2(T = 0.5) \) and \( p = -10(\alpha T = 0.1) \) and thus \( \alpha = \frac{1}{5} \) then \( \omega_{\text{max}} = \sqrt{2.10} = 4.47 \text{ rad/sec.} \)

For \( \alpha = 5, \alpha_{\text{max}} = 40^\circ. \)

**Rule of Thumb:** A maximum of 60° PM should be expected from a lead compensator. If more phase lead is desired then a double lead compensator could be used

\[ D(s) = \left( \frac{Ts + 1}{2Ts + 1} \right)^2 \]

**Example: Lead Compensator for a DC-motor**

Design a lead compensator for \( G(s) = \frac{1}{s(s+1)} \) so the

1) steady state error to a range < 0.1
2) overshoot \( M_p < 25\% \)

Solution: The steady-state error is

\[ e_{ss} = \lim_{s \to 0} s \cdot \frac{1}{1 + KD(s)G(s) \cdot R(s)} \]

with \( R(s) = 1/s^2 \) we get

\[ e_{ss} = \lim_{s \to 0} \left[ \frac{1}{s + KD(s) \cdot 1/s + 1} \right] = \frac{1}{KD(0)}. \]

For \( e_{ss} \leq 0.1 \) and \( KD(s) = K \frac{Ts + 1}{\alpha Ts + 1} \) implies \( K \geq 1.0. \) Therefore take \( K = 10. \)

From Figure 6.37
we that $PM \approx 45^\circ$. The frequency response of $KG(s)$ shows that $pm = 20^\circ$ so we need at least $25^\circ$ more phase margin. To be safe we will design the compensator to add $\sim 40^\circ$ since the compensator also moves the crossover (i.e., changes the magnitude).

In Figure 6.53

we see that $\frac{1}{\alpha} = 5$ gives $40^\circ PM$. So far we have $KD(1) = 10\frac{TS+1}{T/5S+1}$. From the diagram we want in between one and ten, i.e., $\frac{1}{10} < T < 1$. Choose $T = \frac{1}{2}$.

**Design Procedure:**

1. Find low frequency gain $K$ to satisfy the steady state error requirement
2. Select the lead ratio $1/\alpha$ and zero value $1/T$ to give acceptance $PM$ at crossover.
3. The pole location is then $1/\alpha s$.

In lead-compensation there are three primary design parameters:

1. 1crossover frequency $\omega_c$, which determines bandwidth $\omega_B$, rise time $t_r$ and settling time $t_s$.
2. 2Phase margin $PM$, which determines the damping ratio $\zeta$ and overshoot $M_p$.
3. 3The low frequency gain, which determines steady-state error.

A lead compensator increases $\frac{\omega_c}{L(0)} = \left(\frac{\omega_c}{K_r} \text{ for a type 1 system}\right)$.

So low frequency gain and crossover cannot be adjusted independently.

**Lead Compensator Procedure**

1. Determine open-loop gain $K$ to satisfy error or bandwidth requirements:
   - (a) to meet error requirement, pick $K$ to satisfy error constants ($K_p$, $K_v$, or $K_a$) so that $e_{ss}$ error specification is met, or alternatively,
   - (b) to meet bandwidth requirement, pick $K$ so that the open-loop crossover frequency is a factor of two below the desired closed-loop bandwidth.
2. Evaluate the phase margin (PM) of the uncompensated system using the value of $K$ obtained from Step 1.

3. Allow for extra margin about $10^\circ$, and determine the needed phase lead $\phi_{\text{max}}$.

4. Determine $\alpha$ from Equation (6.40) or Figure 6.53.

5. Pick $\omega_{\text{max}}$ to be at the crossover frequency; thus the zero is at $1/T = \omega_{\text{max}}\sqrt{\alpha}$ and the pole is at $1/\alpha T = \omega_{\text{max}}\sqrt{\alpha}$.

6. Draw the compensated frequency response and check the PM.

7. Iterate on the design. Adjust compensator parameters (poles, zeros, and gain) until all specifications are met. Add an additional lead compensator (that is, a double lead compensation) if necessary.

If steady state error conditions cannot be met then we may add a lag-compensator ($\sim$ PI)

$$D(s) = \alpha \frac{Ts + 1}{\alpha Ts + 1}, \quad \alpha > 1.$$  

(we assume the gain $K$ is specified elsewhere)

The typical objective of a lag compensator is to increase the low frequency gain to improve the steady-state characteristics.

Since phase lag is undesirable we should choose $T$ large, i.e., the pole/zero lower than the uncompensated crossover.

**Design Procedure for Lag Compensation**

1. Determine the open-loop gain $K$ that will meet the phase-margin-requirement without compensation.

2. Draw the Bode plot of the uncompensated system with crossover frequency from Step 1, and evaluate the low-frequency gain.

3. Determine $\alpha$ to meet the low frequency gain error requirement.

4. Choose the corner frequency $\omega = 1/T$ (the zero of the lag compensate), to be one octave to one decade below the new crossover frequency $\omega$.

5. The other corner frequency (the pole location of the lag compensator) is then $\omega = 1/\alpha T$.

6. Iterate on the design. Adjust compensator parameters (poles, zeros, gain) to meet all the specifications.

**Example: Lag Compensator for the DC-motor**

A lag compensator is of the form

$$D(s) = K \cdot \frac{s + z}{s + p}; \quad p < z$$

or

$$D(s) = \frac{Ts + 1}{\alpha Ts + 1}, \quad \alpha > 1.$$  

A lag compensator approximates a PI control.