ECE 486 PRACTICE MIDTERM EXAM 1 SOLUTIONS

1. The pendulum dynamics derived in class are

$$\ddot{\theta} = -\frac{g}{\ell}\sin\theta + \frac{1}{m\ell^2}T_e$$

where θ is the angle between the pendulum and the downward vertical direction, g is the gravitational constant, ℓ is the length of the pendulum, m is the tip mass, and T_e is the external torque.

Linearize the above pendulum equation around the upward equilibrium $\theta = \pi$. Write your answer in state space form $\dot{x} = Ax + Bu$ where x is an appropriate vector of state variables and A, B are matrices/vectors of appropriate dimensions.

Solution: To convert to a system of two first-order ODES, let $v := \dot{\theta}$ be angular velocity, then we get

$$\begin{aligned}\dot{\theta} &= v \\ \dot{v} &= -\frac{g}{\ell}\sin\theta + \frac{1}{m\ell^2}T_e\end{aligned}$$

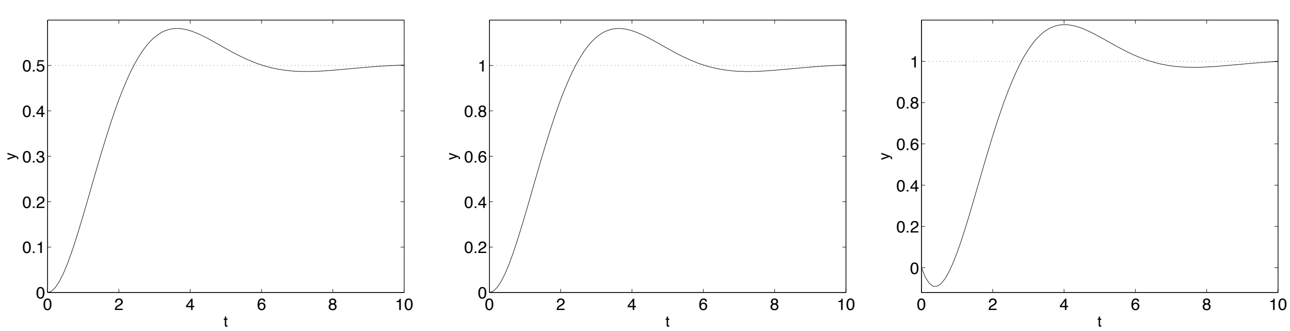
We need a linear approximation of the function $f(\theta) = \sin\theta$ around $\theta = \pi$. From Taylor series, $f(\theta) \approx f(\pi) + f'(\pi)(\theta - \pi) = \sin\pi + \cos\pi \cdot (\theta - \pi) = -(\theta - \pi)$. Taking the state variables to be $\theta - \pi$ (deviation of θ from the equilibrium value π) and v (derivative of $\theta - \pi$, which is the same as derivative of θ), we get the equations in state space form:

$$\frac{d}{dt}\begin{pmatrix}\theta - \pi \\ v\end{pmatrix} = \begin{pmatrix}0 & 1 \\ \frac{g}{\ell} & 0\end{pmatrix}\begin{pmatrix}\theta - \pi \\ v\end{pmatrix} + \begin{pmatrix}0 \\ \frac{1}{m\ell^2}\end{pmatrix}T_e$$

2. Consider the transfer function

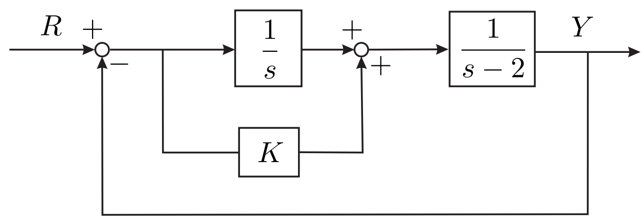
$$H(s) = \frac{1}{s^2 + s + 1}$$

Which of the following is the corresponding step response? Explain your choice and why you rejected the other two possibilities.



Solution: The DC gain of $H(s)$ equals 1, so the left plot is immediately rejected because it corresponds to DC gain of 0.5. The right plot show the correct DC gain, but it has an initial dip which indicates the presence of a RHP zero, while $H(s)$ has no zeros. The middle plot is consistent with a stable second-order transfer function with DC gain of 1 and with no zeros, so it is the correct one.

3. Consider the system given by the block diagram below:



a) Compute the transfer function from the reference R to the output Y.

Solution: Recognizing that the parallel connection on the left side of the diagram has transfer function $\frac{1}{s} + K = \frac{Ks+1}{s}$ and using the formula "forward gain over (one plus loop gain)", we get after some simplifications that

$$\frac{Y}{R} = \frac{Ks+1}{s^2 + (K-2)s + 1}$$

b) Determine the range of values of K for which the system is stable.

Solution: For quadratic polynomials, the necessary condition for stability (all coefficients positive) is also sufficient (as we know from Routh criterion). So, we need $K > 2$.

c) Suppose that the reference is a step: $r(t) = 1(t)$. Does the system achieve perfect steady-state tracking of this reference? If yes, justify. If not, characterize the steady-state tracking error.

Solution: The DC gain equals 1, so yes, we do get perfect tracking. (This is of course true only when the system is stable, that is, for $K > 2$.)

d) Suppose that the reference is a ramp: $r(t) = t1(t)$. Answer the same questions as in part c).

Here we need to apply the final value theorem (FVT) and we need to look at the tracking error $e = r - y$ instead of at y. The relevant transfer function is

$$\frac{E}{R} = \frac{1}{1 + \frac{Ks+1}{s}\frac{1}{s-2}} = \frac{s^2 - 2s}{s^2 + (K-2)s + 1}$$

By FVT, we need to multiply by $1/s^2$ (Laplace transform of the ramp), then multiply by s, and evaluate the result at $s = 0$. This gives

$$\left. \frac{s-2}{s^2 + (K-2)s + 1} \right|_{s=0} = -2$$

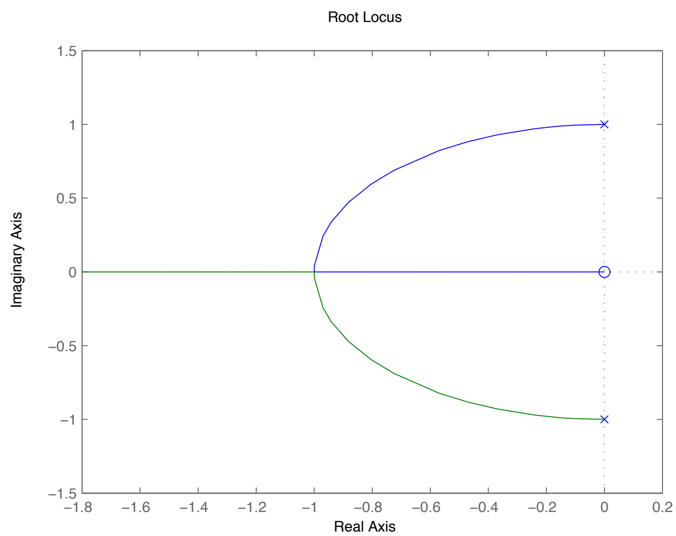
So, the steady-state error is -2, and we don't get perfect tracking.

4. Draw the positive root locus for the transfer function

$$L(s) = \frac{s}{s^2 + 1}$$

Make your sketch as accurate as you can, relying on the rules learned in class. Explain how you used each rule.

Solution:



Open-loop poles are at $\pm j$, open-loop zero is at 0.

Negative real semi-axis is on the root locus. It is the only asymptote.

Departure angles of the branches from the poles are 180°.

Characteristic polynomial $s^2 + Ks + 1$ is stable for all $K > 0$, hence no $j\omega$-axis crossings.

Multiple roots: $s^2 + 1 = 2s^2$, gives $s = \pm 1$, of which only $s = -1$ is on the root locus. This is the "break-in" point. Angle separation at $s = -1$ is 180°, hence by symmetry the two branches arrive vertically, and they depart horizontally.

Furthermore, the break-in point $s = -1$ corresponds to the characteristic polynomial $(s+1)^2 = s^2s+1$, from which we see that the corresponding value of the gain is $K = 2$. For $K \geq 2$ the closed-loop poles are real, which means that the system is perfectly damped.