Plan of the Lecture

- **Review:** Proportional-Integral-Derivative (PID) control
- **Today’s topic:** introduction to Root Locus design method

*Goal:* introduce the Root Locus method as a way of visualizing the locations of closed-loop poles of a given system as some parameter is varied.

*Reading:* FPE, Chapter 5

*Note!!* The way I teach the Root Locus differs a bit from what the textbook does (good news: it is simpler). Still, *pay attention in class!!*
Course structure so far:

modeling — examples
↓
analysis — transfer function, response, stability
↓
design — some simple examples given

We will focus on design from now on.
The Root Locus Design Method
(invented by Walter R. Evans in 1948)

Consider this unity feedback configuration:

\[ R \rightarrow + \rightarrow K \rightarrow L(s) \rightarrow Y \]

where

- \( K \) is a constant gain
- \( L(s) = \frac{b(s)}{a(s)} \), where \( a(s) \) and \( b(s) \) are some polynomials
The Root Locus Design Method

Closed-loop transfer function: \[ \frac{Y}{R} = \frac{KL(s)}{1 + KL(s)}, \quad L(s) = \frac{b(s)}{a(s)} \]

Closed loop poles are solutions of:

\[ 1 + KL(s) = 0 \quad \Leftrightarrow \quad L(s) = -\frac{1}{K} \]

\[ 1 + \frac{Kb(s)}{a(s)} = 0 \]

\[ a(s) + Kb(s) = 0 \quad \text{characteristic equation} \]
A Comment on Change of Notation

Note the change of notation:

\[ H(s) \text{ or } G(s) = \frac{q(s)}{p(s)} \quad \text{to} \quad L(s) = \frac{b(s)}{a(s)} \]

— the RL method is quite general, so \( L(s) \) is not necessarily the *plant* transfer function, and \( K \) is not necessary *feedback gain* (could be *any parameter*).

E.g., \( L(s) \) and \( K \) may be related to plant transfer function and feedback gain through some transformation.

As long as we can represent the poles of the closed-loop transfer function as roots of the equation \( 1 + KL(s) = 0 \) for *some choice* of \( K \) and \( L(s) \), we can apply the RL method.
Towards Quantitative Characterization of Stability

Qualitative description of stability: Routh test gives us a range of $K$ to guarantee stability.

For what values of $K$ do we best satisfy given design specs?
Root Locus and Quantitative Stability

Closed-loop transfer function: \[ \frac{Y}{R} = \frac{KL(s)}{1 + KL(s)}, \quad L(s) = \frac{b(s)}{a(s)} \]

For what values of \( K \) do we best satisfy given design specs?

Specs are encoded in pole locations, so:

The *root locus* for \( 1 + KL(s) \) is the set of all closed-loop poles, i.e., the roots of

\[ 1 + KL(s) = 0, \]

as \( K \) varies from 0 to \( \infty \).
A Simple Example

\[ L(s) = \frac{1}{s^2 + s} \quad b(s) = 1, \quad a(s) = s^2 + s \]

Characteristic equation: \[ a(s) + K b(s) = 0 \]

\[ s^2 + s + K = 0 \]

Here, we can just use the quadratic formula:

\[ s = -\frac{1 \pm \sqrt{1 - 4K}}{2} = -\frac{1}{2} \pm \frac{\sqrt{1 - 4K}}{2} \]

Root locus = \( \left\{ -\frac{1}{2} \pm \frac{\sqrt{1 - 4K}}{2} : 0 \leq K < \infty \right\} \subset \mathbb{C} \)
Example, continued

\[
\text{Root locus } = \left\{ -\frac{1}{2} \pm \frac{\sqrt{1-4K}}{2} \right\} : 0 \leq K < \infty \subset \mathbb{C}
\]

Let’s plot it in the s-plane:

- start at \( K = 0 \quad \text{the roots are } -\frac{1}{2} \pm \frac{1}{2} \equiv -1, 0 \\
note: these are poles of \( L \) (open-loop poles)
Example, continued

Root locus: \( \left\{ -\frac{1}{2} \pm \frac{\sqrt{1-4K}}{2} : 0 \leq K < \infty \right\} \subset \mathbb{C} \)

- as \( K \) increases from 0, the poles start to move

\[
1 - 4K > 0 \quad \implies \quad 2 \text{ real roots}
\]
\[
K = \frac{1}{4} \quad \implies \quad 1 \text{ real root } s = -\frac{1}{2}
\]
Example, continued

Root locus: \[ \left\{ -\frac{1}{2} \pm \frac{\sqrt{1 - 4K}}{2} : 0 \leq K < \infty \right\} \subset \mathbb{C} \]

- as \( K \) increases from 0, the poles start to move

\[ K > \frac{1}{4} \quad \implies \quad 2 \text{ complex roots with } \text{Re}(s) = -\frac{1}{2} \]

\((s = -1/2 \text{ is the point of breakaway from the real axis)}\)
Example, continued

Compare this to admissible regions for given specs:

\[ t_s \approx \frac{3}{\sigma} \quad \text{want } \sigma \text{ large, can only have } \sigma = \frac{1}{2} \quad (t_s = 6) \]

\[ t_r \approx \frac{1.8}{\omega_n} \quad \text{want } \omega_n \text{ large } \implies \text{want } K \text{ large} \]

\[ M_p \quad \text{want to be inside the shaded region } \implies \text{want } K \text{ small} \]

\[ \text{increase } K \]
Thus, the root locus helps us *visualize the trade-off* between all the specs in terms of $K$.

However, for order $> 2$, there will generally be no direct formula for the closed-loop poles as a function of $K$.

**Our goal:** develop simple rules for (approximately) sketching the root locus in the general case.
Equivalent Characterization of RL: Phase Condition

Recall our original definition: The root locus for $1 + KL(s)$ is the set of all closed-loop poles, i.e., the roots of

$$1 + KL(s) = 0,$$

as $K$ varies from 0 to $\infty$.

A point $s \in \mathbb{C}$ is on the RL if and only if

$$L(s) = -\frac{1}{K}$$

for some $K > 0$

negative and real

This gives us an equivalent characterization:

**The phase condition:** The root locus of $1 + KL(s)$ is the set of all $s \in \mathbb{C}$, such that $\angle L(s) = 180^\circ$, i.e., $L(s)$ is real and negative.
Six Rules for Sketching Root Loci

There are *six rules* for sketching root loci. These rules are mainly qualitative, and their purpose is to give intuition about impact of poles and zeros on performance.

These rules are:

- Rule A — number of branches
- Rule B — start points
- Rule C — end points
- Rule D — real locus
- Rule E — asymptotes
- Rule F — $j\omega$-crossings

Today, we will cover mostly Rules A–C (and a bit of D).
Rule A: Number of Branches

\[
1 + K \frac{b(s)}{a(s)} = 1 + K \frac{s^m + b_1 s^{m-1} + \ldots + b_{m-1} s + b_m}{s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n} = 0
\]

\[
\implies (s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n) + K (s^m + b_1 s^{m-1} + \ldots + b_{m-1} s + b_m) = 0
\]

Since \( \text{deg}(a) = n \geq m = \text{deg}(b) \), the characteristic polynomial \( a(s) + Kb(s) = 0 \) has degree \( n \).

The characteristic polynomial has \( n \) solutions (roots), some of which may be repeated. As we vary \( K \), these \( n \) solutions also vary to form \( n \) branches.

**Rule A:**

\[
\#(\text{branches}) = \text{deg}(a)
\]
The locus starts from $K = 0$. What happens near $K = 0$?

If $a(s) + Kb(s) = 0$ and $K \sim 0$, then $a(s) \approx 0$.

Therefore:

- $s$ is close to a root of $a(s) = 0$, or
- $s$ is close to a pole of $L(s)$

**Rule B:** branches start at open-loop poles.
Rule C: End Points

What happens to the locus as $K \to \infty$?

$$a(s) + Kb(s) = 0$$

$$b(s) = -\frac{1}{K}a(s)$$

— as $K \to \infty$,

- branches end at the roots of $b(s) = 0$, or
- branches end at zeros of $L(s)$

**Rule C:** branches end at open-loop zeros.

**Note:** if $n > m$, we have $n$ branches, but only $m$ zeros. The remaining $n - m$ branches go off to infinity (end at “zeros at infinity”).
Example
PD control of an unstable 2nd-order plant

\[
\begin{align*}
\frac{Y}{R} &= \frac{G_c G_p}{1 + G_c G_p} \\
\text{poles: } 1 + G_c(s)G_p(s) &= 0 \\
1 + (K_P + K_D s) \left( \frac{1}{s^2 - 1} \right) &= 0
\end{align*}
\]

We will examine the impact of varying \( K = K_D \), assuming the ratio \( K_P / K_D \) fixed.
Example

PD control of an unstable 2nd-order plant

We will examine the impact of varying $K = K_D$, assuming the ratio $K_P/K_D$ fixed.

Let us write the characteristic equation in *Evans form*:

$$1 + \frac{K_D}{K} \left( s + \frac{K_P}{K_D} \right) \left( \frac{1}{s^2 - 1} \right) = 1 + \frac{K \left( s + \frac{K_P}{K_D} \right)}{s^2 - 1} = 0$$

$$L(s) = \frac{s - z_1}{s^2 - 1} \quad \text{zero at} \quad s = z_1 = -\frac{K_P}{K_D} < 0$$
Example

\[ L(s) = \frac{s - z_1}{s^2 - 1} \]

- **Rule A:** \[ \begin{cases} m = 1 \\ n = 2 \end{cases} \implies 2 \text{ branches} \]
- **Rule B:** branches start at open-loop poles, \[ s = \pm 1 \]
- **Rule C:** branches end at open-loop zeros, \[ s = z_1, -\infty \]
  (we will see why \(-\infty\) later)

So the root locus will look something like this:
Why does one of the branches go off to $-\infty$?

\[ s^2 - 1 + K(s - z_1) = 0 \]
\[ s^2 + Ks - (Kz_1 + 1) = 0 \]

\[ s = -\frac{K}{2} \pm \sqrt{\frac{K^2}{4} + Kz_1 + 1}, \quad z_1 < 0 \]

as $K \to \infty$, $s$ will be $< 0$
Is the point \( s = 0 \) on the root locus?

Let’s see if there is any value \( K > 0 \), for which this is possible:

\[
1 + KL(0) = 0
\]

\[
1 + Kz_1 = 0 \quad K = -\frac{1}{z_1} > 0 \text{ does the job}
\]
From Root Locus to Time Response Specs

For concreteness, let’s see what happens when

\[ KP/K_D = -z_1 = 2 \quad \text{and} \quad K = K_D = 5 \implies KP = 10 \]

Characteristic equation:

\[
1 + 5 \left( \frac{s + 2}{s^2 - 1} \right) = 0
\]

\[ s^2 + 5s + 9 = 0 \]

Relate to 2nd-order response:

\[ \omega_n^2 = 9, \ 2\zeta \omega_n = 5 \implies \zeta = 5/6 \]
Main Points

- When zeros are in LHP, \textit{high gain} can be used to stabilize the system (although one must worry about zeros at infinity).
- If there are zeros in RHP, high gain is always disastrous.
- PD control is effective for stabilization because it introduces a zero in LHP.

\textbf{But:} Rules A–C cannot tell the whole story. How do we know which way the branches go, and which pole corresponds to which zero?

\textbf{Rules D–F!!}
Example

Let’s consider

\[ L(s) = \frac{s + 1}{s(s + 2)(s + 1)^2 + 1} \]

- **Rule A:** \( m = 1 \), \( n = 4 \) \( \implies \) 4 branches

- **Rule B:** branches start at open-loop poles
  
  \[ s = 0, s = -2, s = -1 \pm j \]

- **Rule C:** branches end at open-loop zeros
  
  \[ s = -1, \pm \infty \]
Example, continued

Three more rules:

- Rule D: real locus
- Rule E: asymptotes
- Rule F: \( j\omega \)-crossings

Rules D and E are both based on the fact that

\[
1 + KL(s) = 0 \text{ for some } K > 0 \iff L(s) < 0
\]
Rule D: Real Locus

The branches of the RL start at the open-loop poles. Which way do they go, left or right?

Recall the phase condition:

\[ 1 + KL(s) = 0 \iff \angle L(s) = 180^\circ \]

\[ \angle L(s) = \angle \frac{b(s)}{a(s)} \]

\[ = \angle \frac{(s - z_1)(s - z_2) \ldots (s - z_m)}{(s - p_1)(s - p_2) \ldots (s - p_n)} \]

\[ = \sum_{i=1}^{m} \angle (s - z_i) - \sum_{j=1}^{n} \angle (s - p_j) \]

— this sum must be \( \pm 180^\circ \) for any \( s \) that lies on the RL.
Rule D: Real Locus

So, we try test points:

\[ \angle(s_1 - z_1) = 0^\circ \quad (s_1 > z_1) \]
\[ \angle(s_1 - p_1) = 180^\circ \quad (s_1 < p_1) \]
\[ \angle(s_1 - p_2) = 0^\circ \quad (s_1 > p_2) \]
\[ \angle(s_1 - p_3) = -\angle(s_1 - p_4) \]
(conjugate poles cancel)

\[ \angle(s_1 - z_1) - [\angle(s_1 - p_1) + \angle(s_1 - p_2) + \angle(s_1 - p_3) + \angle(s_1 - p_4)] \]
\[ = 0^\circ - [180^\circ + 0^\circ + 0^\circ] = -180^\circ \quad \checkmark s_1 \text{ is on RL} \]
Rule D: Real Locus

Try more test points:

\[ \angle(s_2 - z_1) = 180^\circ \quad (s_2 < z_2) \]
\[ \angle(s_2 - p_1) = 180^\circ \quad (s_2 < p_1) \]
\[ \angle(s_2 - p_2) = 0^\circ \quad (s_2 > p_2) \]
\[ \angle(s_2 - p_3) = -\angle(s_1 - p_4) \]
(conjugate poles cancel)

\[ \angle(s_2 - z_1) - [\angle(s_2 - p_1) + \angle(s_2 - p_2) + \angle(s_2 - p_3) + \angle(s_2 - p_4)] = 180^\circ - [180^\circ + 0^\circ + 0^\circ] = 0^\circ \quad \times \text{s}_1 \text{ is not on RL} \]
Rule D: Real Locus

Rule D: If \( s \) is real, then it is on the RL of \( 1 + KL \) if and only if there are an odd number of real open-loop poles and zeros to the right of \( s \).

We will cover Rules E and F, and complete the RL for this example, in the next lecture.