Reading: FPE, Section 3.3–3.4

Consider a system with transfer function of the form

$$H(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}.$$  

The degree \((n)\) of the denominator polynomial is called the order of the system. As we have already seen, systems of order one and two arise frequently in practice; even when the order of a system is higher than two, one can sometimes approximate the system by a first or second order system in order to obtain intuition about the system’s behavior. It is therefore important for us to study the responses of first and second order systems to various types of inputs (and in particular, to step inputs).

Note that if all poles of \(H(s)\) are in the OLHP, the Final Value Theorem states that the steady-state value of the step response is given by

$$\lim_{s \to 0} s Y(s) = \lim_{s \to 0} s H(s) \frac{1}{s} = \lim_{s \to 0} H(s) = H(0).$$

The quantity \(H(0)\) is called the DC gain of the system. This is true regardless of the order of the system.

1 Step Response of First Order Systems

Consider a first order system with transfer function

$$H(s) = \frac{b_0}{s + a_0}, \quad a_0 \neq 0.$$  

This system has a single pole at \(s = -a_0\). The step response of this system is obtained by calculating

$$Y(s) = H(s) U(s) = \frac{b_0}{s + a_0} \frac{1}{s} = \frac{b_0}{a_0} \left( \frac{1}{s} - \frac{1}{s + a_0} \right),$$

which produces

$$y(t) = \frac{b_0}{a_0} 1(t) - \frac{b_0}{a_0} e^{-a_0 t}, \quad t \geq 0.$$  

Consider two cases:

- If \(a_0 > 0\), then the pole of the transfer function is in the OLHP, and \(e^{-a_0 t} \to 0\) as \(t \to \infty\). The step response thus reaches a steady-state value of \(\frac{b_0}{a_0}\) (which is the DC gain of the system). The response is said to be stable.

- If \(a_0 < 0\), the pole of the transfer function is in the ORHP, and \(e^{-a_0 t} \to \infty\) as \(t \to \infty\). The step response therefore goes to \(\infty\), and there is no steady state value. The response is said to be unstable.

**Question:** What is a measure of performance that can be used to evaluate the step-response?

**Potential answer:** The amount of time taken to get close to its final value.

**Rise time** \((t_r)\). The rise time of a step response is defined as the amount of time required for the response to go from 10% of its final value to 90% of its final value.

The rise time of a first order system with transfer function \(H(s) = \frac{b_0}{s + a_0}\) is \(t_r = \frac{\ln 9}{a_0}\) (we will prove this in a homework problem).
2 Step Response of Second Order Systems

Consider a second order system of the form

$$H(s) = \frac{b_1 s + b_0}{s^2 + a_1 s + a_0}.$$ 

In order to obtain intuition about these systems, we will be focusing on a particular form of second order system:

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta \omega_n s + \omega_n^2}, \quad \omega_n > 0.$$ 

The poles of this transfer function are obtained from the quadratic formula as $s = -\zeta \omega_n \pm \omega_n \sqrt{\zeta^2 - 1}$. The location of these poles in the complex plane will vary based on the value of $\zeta$. We analyze three different cases:

- $0 \leq \zeta < 1$: The system has two complex poles in the CLHP (they will be in the OLHP if $\zeta > 0$). The system is said to be underdamped.
- $\zeta = 1$: The system has two repeated poles at $s = -\omega_n$. The system is said to be critically damped.
- $\zeta > 1$: The system has two poles on the negative real axis. The system is said to be overdamped.
2.1 Underdamped and Critically Damped Systems \((0 \leq \zeta \leq 1)\)

The poles of the transfer function in this case are given by \(s = -\zeta \omega_n \pm j\omega_n \sqrt{1-\zeta^2}\). To simplify the notation, define

\[
\sigma = \zeta \omega_n, \quad \omega_d = \omega_n \sqrt{1-\zeta^2},
\]

in which case the poles are at \(s = -\sigma \pm j\omega_d\). The transfer function can be written as

\[
H(s) = \frac{\omega_n^2}{(s + \sigma + j\omega_d)(s + \sigma - j\omega_d)} = \frac{\omega_n^2}{(s + \sigma)^2 + \omega_d^2}.
\]

The Laplace Transform of the step response is given by \(Y(s) = H(s)\frac{1}{s}\), and using a Laplace Transform table (entry 21 in FPE), we see that

\[
y(t) = 1 - e^{-\sigma t} \left( \cos \omega_d t + \frac{\sigma}{\omega_d} \sin \omega_d t \right). \tag{1}
\]

- When \(\zeta = 0\), we have \(\sigma = 0\) and \(\omega_d = \omega_n\), and the response becomes \(y(t) = 1 - \cos \omega_n t\), which oscillates between 0 and 2 for all \(t\).
- When \(\zeta = 1\), we have \(\sigma = \omega_n\) and \(\omega_d = 0\), and the response becomes \(y(t) = 1 - e^{-\omega_n t}(1 + \omega_n t)\); one can verify that this does not have any oscillations at all, and asymptotically reaches the steady state value of 1 (this is the DC gain \(H(0)\)).

The response for intermediate values of \(\zeta\) falls in between these two extremes. The behavior of \(y(t)\) for different values of \(\zeta\) (with a fixed value of \(\omega_n\)) is shown in Fig. 3.

As we can see from the responses for various values of \(\zeta\), a larger value of \(\zeta\) corresponds to fewer oscillations and less overshoot, and thus \(\zeta\) is said to represent the amount of damping in the response (a higher value of \(\zeta\) corresponds to more damping). \(\zeta\) is called the damping ratio. Examining Fig. 2, we see that the angle between the imaginary axis and the vector connecting the pole to the origin is given by \(\sin^{-1} \zeta\).
The magnitude of the vector is given by $\omega_n$. Thus, as $\zeta$ increases with a fixed $\omega_n$, the poles move on the perimeter of a circle toward the real axis.

The quantity $\omega_n$ is called the **undamped natural frequency**, since it represents the location of the pole on the imaginary axis when there is no damping (i.e., when $\zeta = 0$). The quantity $\omega_d$ is called the **damped natural frequency**, since it represents the imaginary part of the pole when there is damping. When $\zeta = 0$, we have $\omega_d = \omega_n$. If $\zeta > 0$ and we increase $\omega_n$, the poles move further to the left in the OLHP. This causes the term $\sigma = \xi \omega_n$ to increase, which causes the term $e^{-\sigma t}$ to die out faster in the step response (equation (1)). Thus, increasing $\omega_n$ has the effect of making the oscillations die out faster (and, in general, making the system respond faster).

### 2.2 Overdamped System ($\zeta > 1$)

The poles of the transfer function in this case are given by $s = -\zeta \omega_n \pm \omega_n \sqrt{\zeta^2 - 1}$, which are both on the negative real axis (as $\zeta \to \infty$, one pole approaches $s = 0$, and the other pole approaches $s = -\infty$). The transfer function can be written as

$$H(s) = \frac{\omega_n^2}{(s + \zeta \omega_n + \omega_n\sqrt{\zeta^2 - 1})(s + \zeta \omega_n - \omega_n\sqrt{\zeta^2 - 1})}.$$

The Laplace Transform of the step response is given by $Y(s) = H(s) \frac{1}{s}$, and using partial fraction expansion, we see that the step response will have the form

$$y(t) = 1 - k_1 e^{(-\zeta \omega_n - \omega_n \sqrt{\zeta^2 - 1})t} - k_2 e^{(-\zeta \omega_n + \omega_n \sqrt{\zeta^2 - 1})t},$$

for some constants $k_1$ and $k_2$. This response has no oscillations; a sample response is shown in Fig. 4

![Figure 4: Step response of an overdamped system ($\zeta > 1$)](image)

### 2.3 Discussion

Based on the above analysis, we can come to the following general conclusions about how the poles of second order systems affect their step responses.

- If the poles are complex, the step response will have oscillations and overshoot.
- As the poles move toward the real axis while maintaining a fixed distance from the origin, the amount of oscillation decreases (leading to less overshoot) – this corresponds to the damping ratio $\zeta$ increasing (with a fixed $\omega_n$),
• If $\omega_n$ increases, the poles move further left in the OLHP, and the oscillations die out faster.
• If all poles are on the negative real axis, there will be no oscillation and no overshoot.

Note that if the transfer function has one or more poles in the open right half plane, the step response will contain a term that goes to $\infty$ as $t \to \infty$, and thus there is no steady state response.

Based on these general rules, the step responses for various pole locations are shown in Fig. 5.

Figure 5: Step Responses Associated with Locations of Poles in the Complex Plane.