1.3 Bode Plot for \( \left( \frac{j\omega}{\omega_n} \right)^2 + 2\zeta\left( \frac{\omega}{\omega_n} \right)j + 1 \)^\pm 1

We have already seen what the magnitude and phase plots look like for a second order system of this form. To derive general rules for drawing this, note that the magnitude of this function at \( s = j\omega \) is given by

\[
\log \left| \left( \frac{j\omega}{\omega_n} \right)^2 + 2\zeta\left( \frac{\omega}{\omega_n} \right)j + 1 \right| = \log \sqrt{\left(1 - \left( \frac{\omega}{\omega_n} \right)^2 \right)^2 + 4\zeta^2 \left( \frac{\omega}{\omega_n} \right)^2}.
\]

For \( \omega \ll \omega_n \), we have \( \frac{\omega}{\omega_n} \approx 0 \), and so \( \log \left| \left( \frac{j\omega}{\omega_n} \right)^2 + 2\zeta\left( \frac{\omega}{\omega_n} \right)j + 1 \right| \approx 0 \). For \( \omega \gg \omega_n \), we have

\[
\sqrt{\left(1 - \left( \frac{\omega}{\omega_n} \right)^2 \right)^2 + 4\zeta^2 \left( \frac{\omega}{\omega_n} \right)^2} \approx \sqrt{\left( \frac{\omega}{\omega_n} \right)^4} = \left( \frac{\omega}{\omega_n} \right)^2,
\]

and so \( \log \left| \left( \frac{j\omega}{\omega_n} \right)^2 + 2\zeta\left( \frac{\omega}{\omega_n} \right)j + 1 \right| \approx 2\log \omega - 2\log \omega_n \). This is a line of slope 2 passing through the point 0 when \( \omega = \omega_n \). The general magnitude curve for \( \log \left| \left( \frac{j\omega}{\omega_n} \right)^2 + 2\zeta\left( \frac{\omega}{\omega_n} \right)j + 1 \right| \) thus looks like:

The phase of \( \left( \frac{j\omega}{\omega_n} \right)^2 + 2\zeta\left( \frac{\omega}{\omega_n} \right)j + 1 \) is given by

\[
\angle \left( \frac{j\omega}{\omega_n} \right)^2 + 2\zeta\left( \frac{\omega}{\omega_n} \right)j + 1 = \tan^{-1} \left( \frac{2\zeta \frac{\omega}{\omega_n}}{1 - \left( \frac{\omega}{\omega_n} \right)^2} \right).
\]

For \( \omega \ll \omega_n \), the argument of the tan function is almost 0, and so the phase curve starts at 0 for small \( \omega \). For \( \omega = \omega_n \), the argument is \( \infty \), and so the phase curve passes through \( \frac{\pi}{2} \) when \( \omega = \omega_n \). For \( \omega \gg \omega_n \), the argument of the tan function approaches 0 from the negative side, and so the phase curve approaches \( \pi \) for large values of \( \omega \). Just as in the first order case, we will take the phase curve transitions to occur one decade before and after \( \omega_n \). This produces a phase curve of the form:

The Bode plot for the factor \( \left( \frac{j\omega}{\omega_n} \right)^2 + 2\zeta\left( \frac{\omega}{\omega_n} \right)j + 1 \)^\pm 1 looks just like the Bode plot for the factor
\[ \left(\frac{j \omega}{\omega_n}\right)^2 + 2 \zeta \left(\frac{\omega}{\omega_n}\right) j + 1 \], except that everything is flipped:

**Example.** Draw the Bode Plot of \( H(s) = \frac{(s+1)(s^2+3s+10)}{s^2(s+10)(s+100)} \)

**Solution.**

**Example.** Draw the Bode Plot of \( H(s) = 5 \frac{\frac{s}{50} + 1}{\left(\frac{s}{50}\right)^2 + s \frac{6}{50} + 1} \)

**Solution.**
1.4 Nonminimum Phase Systems

So far we have been looking at the case where all zeros and poles are in the CLHP. Bode plots can also be
drawn for systems that have zeros or poles in the RHP – however, note that for systems that have RHP
poles, the steady state response to a sinusoidal input will not be a sinusoid (there won’t even be a steady
state response, as the output will blow up). This does not change the fact that the transfer function will
have a magnitude and phase at every frequency (since the transfer function is simply a complex number at
every frequency $\omega$). Transfer functions with zeros in the right half plane are called nonminimum phase
systems, and those with all zeros and poles in the CLHP are called minimum phase systems. To see
how the Bode plot of a nonminimum phase system compares to that of a minimum phase system, consider
the following example.

**Example.** Draw the Bode plots for the systems

$$H_1(s) = 10 \frac{s + 1}{s + 10}, \quad H_2(s) = 10 \frac{s - 1}{s + 10}.$$

**Solution.**
As we can see from the above example, the magnitudes of the two transfer functions do not depend on whether the zero is in RHP or the LHP. However, the phase plots are quite different. This brings us to the following fact:

For systems with the same magnitude characteristic, the range in phase angle of the minimum phase system is smaller than the range in phase angle of any nonminimum phase system.

Note that for minimum phase systems, the magnitude plot uniquely determines the transfer function, but for nonminimum phase systems, we need both the magnitude plot and the phase plot in order to determine the transfer function.