# Plan of the Lecture

- **Review:** stability from frequency response
- **Today's topic:** control design using frequency response

*Goal:* understand the effect of various types of controllers (PD/lead, PI/lag) on the closed-loop performance by reading the open-loop Bode plot; develop frequency-response techniques for shaping transient and steady-state response using dynamic compensation

*Reading:* FPE, Chapter 6

# Review: Phase Margin for 2nd-Order System

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s}, \qquad \text{closed-loop t.f.} \;= \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\text{PM}\Big|_{K=1} = \tan^{-1}\left(\frac{2\zeta}{\sqrt{4\zeta^4+1} - 2\zeta^2}\right) \approx 100 \cdot \zeta$$

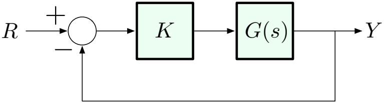
Conclusions:

larger PM $\Longleftrightarrow$ better damping

(open-loop quantity) (closed-loop characteristic)

Thus, the overshoot $M_p = \exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)$ and resonant peak $M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}} - 1$ are both related to PM through $\zeta$!!

# Bode's Gain-Phase Relationship




Assuming that $G(s)$ is *minimum-phase* (i.e., has no RHP zeros), we derived the following for the Bode plot of $KG(s)$:

| | low freq. | real zero/pole | complex zero/pole |
|---|---|---|---|
| mag. slope | $n$ | up/down by 1 | up/down by 2 |
| phase | $n \times 90°$ | up/down by 90° | up/down by 180° |

We can state this succinctly as follows:

**Gain-Phase Relationship.** Far enough from break-points,

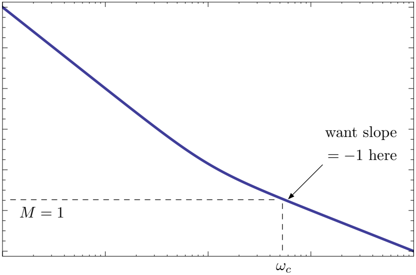
$$\text{Phase} \approx \text{Magnitude Slope} \times 90°$$

# Bode's Gain-Phase Relationship

**Gain-Phase Relationship.** Far enough from break-points,

$$\text{Phase} \approx \text{Magnitude Slope} \times 90^\circ$$

This suggests the following rule of thumb:

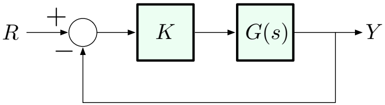



- $M$ has slope $-2$ at $\omega_c$
  $\Rightarrow \phi(\omega_c) = -180^\circ$
  $\Rightarrow$ bad (no PM)
- $M$ has slope $-1$ at $\omega_c$
  $\Rightarrow \phi(\omega_c) = -90^\circ$
  $\Rightarrow$ good (PM $= 90^\circ$)

— this is an important *design guideline*!!

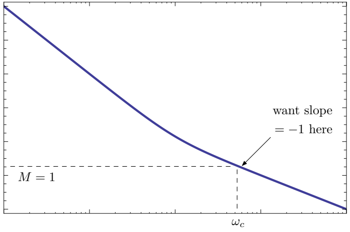
(Similar considerations apply when $M$-plot has positive slope – depends on the t.f.)

# Gain-Phase Relationship & Bandwidth




$$\begin{cases} |KG(j\omega_c)| = 1 \\ \angle G(j\omega_c) = -90^\circ \end{cases} \quad \Rightarrow KG(j\omega_c) = -j$$

M-plot for *open-loop* t.f. $KG$:




Note: $|KG(j\omega)| \to \infty$ as $\omega \to 0$

*Closed-loop* t.f.:

$$T(j\omega_c) = \frac{KG(j\omega_c)}{1 + KG(j\omega_c)} = \frac{-j}{1-j}$$

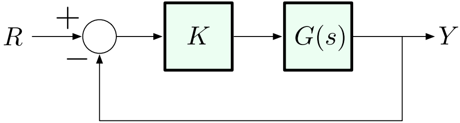
$$|T(j\omega_c)| = \left| \frac{-j}{1-j} \right| = \frac{1}{\sqrt{2}}$$

$$|T(0)| = \lim_{\omega \to 0} \frac{|KG(j\omega)|}{|1 + KG(j\omega)|} = 1$$

$$\Longrightarrow \omega_c = \omega_{\text{BW}} \text{ (bandwidth)}$$

- If PM $= 90^\circ$, then $\omega_c = \omega_{\text{BW}}$
- If $PM < 90^\circ$, then $\omega_c \le \omega_{\text{BW}} \le 2\omega_c$ (see FPE)

# Control Design Using Frequency Response



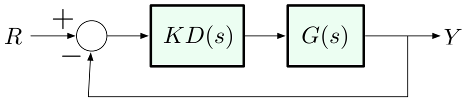

Bode's Gain-Phase Relationship suggests that we can shape the time response of the *closed-loop* system by choosing $K$ (or, more generally, a dynamic controller $KD(s)$) to tune the Phase Margin.

In particular, from the quantitative Gain-Phase Relationship,

$$\text{Magnitude slope}(\omega_c) = -1 \qquad \Longrightarrow \qquad \text{Phase}(\omega_c) \approx -90^\circ$$

— which gives us PM of $90^\circ$ and consequently good damping.

# Example




$$\text{Let } G(s) = \frac{1}{s^2} \qquad \text{(double integrator)}$$

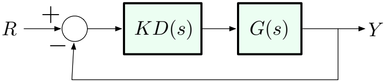
Objective: design a controller $KD(s)$ ($K$ = scalar gain) to give

- stability
- good damping (will make this more precise in a bit)
- $\omega_{\mathrm{BW}} \approx 0.5$ (always a closed-loop characteristic)

Strategy:

- from Bode's Gain-Phase Relationship, we want magnitude slope $= -1$ at $\omega_c \Longrightarrow \mathrm{PM} = 90° \Longrightarrow$ good damping;
- if PM $= 90°$, then $\omega_c = \omega_{\mathrm{BW}} \Longrightarrow$ want $\omega_c \approx 0.5$

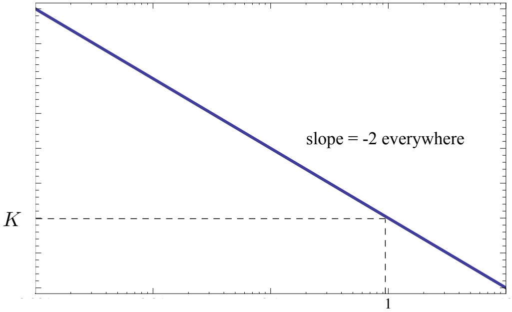
# Design, First Attempt




$$G(s) = \frac{1}{s^2}$$

Let's try proportional feedback:

$$D(s) = 1 \implies KD(s)G(s) = KG(s) = \frac{K}{s^2}$$

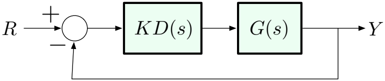



This is not a good idea: slope $= -2$ everywhere, so no PM.

We already know that P-gain alone won't do the job:

$$K + s^2 = 0 \text{ (imag. poles)}$$

# Design, Second Attempt




$$G(s) = \frac{1}{s^2}$$

Let's try proportional-derivative feedback:

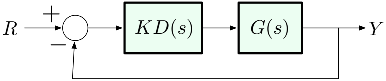
$$KD(s) = K(\tau s + 1), \qquad \text{where } K = K_{\mathrm{P}},\ K\tau = K_{\mathrm{D}}$$

Open-loop transfer function: $KD(s)G(s) = \dfrac{K(\tau s + 1)}{s^2}$.

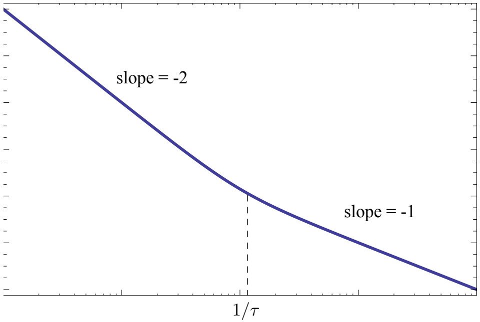
Bode plot interpretation: PD controller introduces a Type 2 term in the numerator, which pushes the slope up by 1

— this has the effect of pushing the M-slope of $KD(s)G(s)$ from $-2$ to $-1$ past the break-point ($\omega = 1/\tau$).

# Design, Second Attempt (PD-Control)




Open-loop transfer function: $KD(s)G(s) = \dfrac{K(\tau s + 1)}{s^2}$




For the G-P relationship to be valid, choose the break-point several times smaller than desired $\omega_c$:

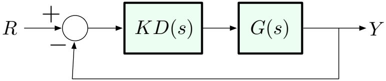
$\Longrightarrow$ let's take $\tau = 10$

$\Longrightarrow \dfrac{1}{\tau} = 0.1 = \dfrac{\omega_c}{5}$

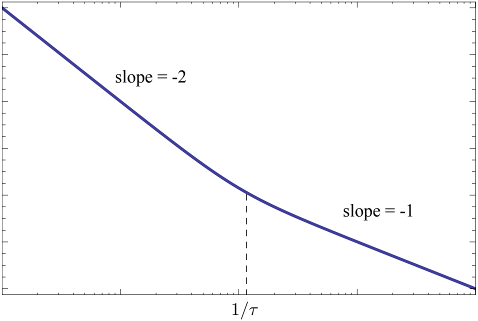
Open-loop t.f.:

$$KD(s)G(s) = \frac{K(10s + 1)}{s^2}$$

# Design, Second Attempt (PD-Control)




Open-loop transfer function: $KD(s)G(s) = \dfrac{K(10s+1)}{s^2}$

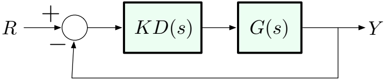



- Want $\omega_c \approx 0.5$
- This means that

$$\begin{aligned} M(j0.5) &= 1 \\ |KD(j0.5)G(j.05)| & \\ &= \frac{K|5j+1|}{0.5^2} \\ &= 4K\sqrt{26} \approx 20K \end{aligned}$$

$$\Longrightarrow K = \frac{1}{20}$$

# PD Control Design: Evaluation




$$G(s) = \frac{1}{s^2}$$

Initial design: $KD(s) = \dfrac{10s + 1}{20}$

What have we accomplished?

- PM $\approx 90^\circ$ at $\omega_c = 0.5$
- still need to check in Matlab and iterate if necessary

Trade-offs:

- want $\omega_{\rm BW}$ to be large enough for fast response (larger $\omega_{\rm BW} \longrightarrow$ larger $\omega_n \longrightarrow$ smaller $t_r$), but not too large to avoid noise amplification at high frequencies
- PD control increases slope $\longrightarrow$ increases $\omega_c \longrightarrow$ increases $\omega_{\rm BW} \longrightarrow$ faster response
- usual complaint: D-gain is not physically realizable, so let's try lead compensation

# Lead Compensation: Bode Plot

$$KD(s) = K\frac{s+z}{s+p}, \qquad p \gg z$$

In Bode form:

$$KD(s) = \frac{Kz\left(\frac{s}{z}+1\right)}{p\left(\frac{s}{p}+1\right)}$$

or, absorbing $z/p$ into the overall gain, we have

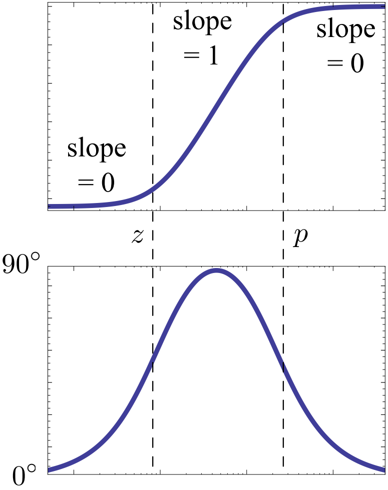
$$KD(s) = \frac{K\left(\frac{s}{z}+1\right)}{\left(\frac{s}{p}+1\right)}$$

## Break-points:

- Type 1 zero with break-point at $\omega = z$ (comes first, $z \ll p$)
- Type 1 pole with break-point at $\omega = p$

# Lead Compensation: Bode Plot

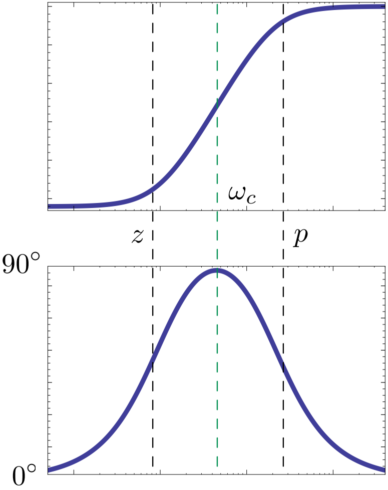
$$KD(s) = \frac{K\left(\frac{s}{z}+1\right)}{\left(\frac{s}{p}+1\right)}$$




- magnitude levels off at high frequencies $\Longrightarrow$ better noise suppression
- adds phase, hence the term "phase lead"

# Lead Compensation and Phase Margin

$$KD(s) = \frac{K\left(\frac{s}{z}+1\right)}{\left(\frac{s}{p}+1\right)}$$




For best effect on PM, $\omega_c$ should be halfway between $z$ and $p$ (on log scale):

$$\log \omega_c = \frac{\log z + \log p}{2}$$

$$\text{or } \omega_c = \sqrt{z \cdot p}$$

— geometric mean of $z$ and $p$

Trade-offs: large $p - z$ means

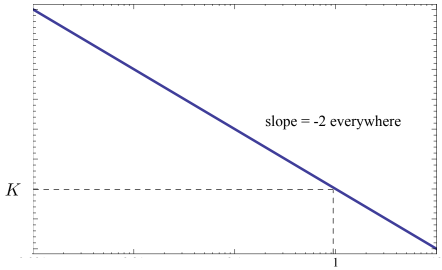
- large PM (closer to 90°)
- but also bigger $M$ at higher frequencies (worse noise suppression)

## Back to Our Example: $G(s) = \frac{1}{s^2}$

Objectives (same as before):

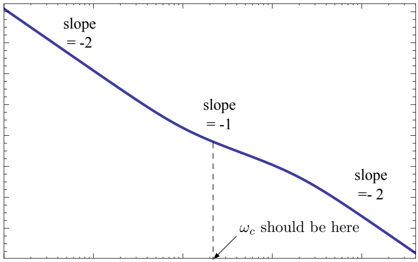
- stability
- good damping
- $\omega_{\mathrm{BW}}$ close to 0.5

$$KG(s) = \frac{K}{s^2} \text{ (w/o lead):}$$



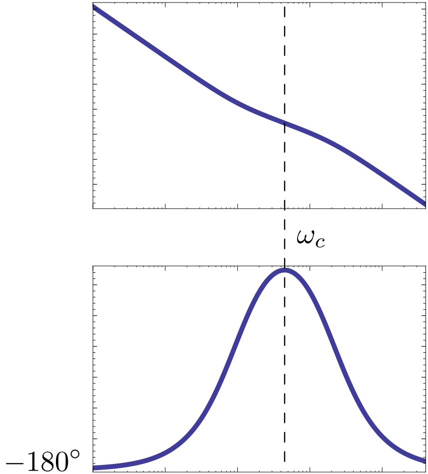

$$\frac{K}{(0.5)^2} = 1 \implies K = \frac{1}{4}$$

after adding lead:




— adding lead will increase $\omega_c$!!

# Back to Our Example: $G(s) = \frac{1}{s^2}$




After adding lead with $K = 1/4$, what do we see?

- adding lead increases $\omega_c$
- $\Longrightarrow$ PM $< 90°$
- $\Longrightarrow$ $\omega_{\text{BW}}$ may be $> \omega_c$

To be on the safe side, we choose a *new value* of $K$ so that

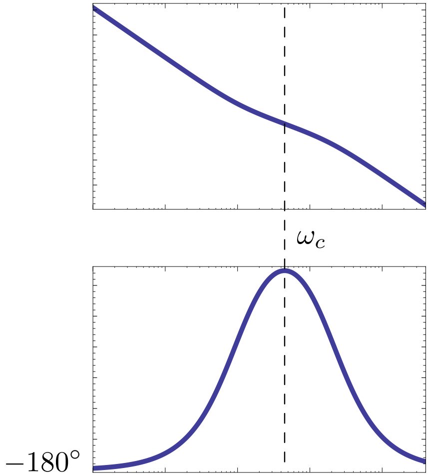
$$\omega_c = \frac{\omega_{\text{BW}}}{2}$$

(b/c generally $\omega_c \le \omega_{\text{BW}} \le 2\omega_c$)

Thus, we want

$$\omega_c = 0.25 \implies K = \frac{1}{16}$$

# Back to Our Example: $G(s) = \frac{1}{s^2}$




Next, we pick $z$ and $p$ so that $\omega_c$ is approximately their geometric mean:

e.g., $z = 0.1$, $p = 2$

$$\sqrt{z \cdot p} = \sqrt{0.2} \approx 0.447$$

Resulting lead controller:

$$KD(s) = \frac{1}{16} \frac{\frac{s}{0.1} + 1}{\frac{s}{2} + 1}$$

(may still need to be refined using Matlab)

# Lead Controller Design Using Frequency Response

## General Procedure

1. Choose $K$ to get desired bandwidth spec w/o lead
2. Choose lead zero and pole to get desired PM
   - in general, we should first check PM with the $K$ from 1, w/o lead, to see how much more PM we need
3. Check design and iterate until specs are met.

This is an intuitive procedure, but it's not very precise, requires trial & error.