ECE 463: Digital Communications Lab.

Lecture 8: Modulation III
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Previous Lecture:

- ✔ CFO Estimation
- ✔ CFO Correction
- ✔ Frame Synchronization
- ✔ Phase Tracking

This Lecture:

- ❌ Maximum Likelihood Decoding
- ❌ QAM & QPSK
- ❌ BER vs. SNR
- ❌ Quantization Noise & AGCs
Digital Communication System

Bits-to-Symbols Mapper (Encoding)

Bits

Pulse Shaping

DAC

LPF

Mixer

BPF

PA

TX

RX

Symbols-to-Bits Mapper (Decoding)

Symbols

ADC

Matched Filter

Symbol Timing Recovery

Frame Sync.

CFO Correction

Channel Equalization

Demodulation

LNA

BPF

LPF

PLL

Mixer

Frame Sync.

CFO Correction

Channel Equalization

Demodulation

Symbols-to-Bits Mapper (Decoding)

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Demodulation

Symbols-to-Bits Mapper (Decoding)
BPSK Modulation

$0 \rightarrow -1$
$1 \rightarrow +1$

$x(t)$
Transmitted Constellation

$y(t) = hx(t) + v(t)$
Received Constellation

After,
Matched Filtering
Timing Recovery
CFO Correction
BPSK Modulation

Transmitted Constellation

Received Constellation

\[
x(t)
\]

\[
y(t) = x(t) + \nu(t)/h
\]

After,
Matched Filtering
Timing Recovery
CFO Correction
Channel Equalization
BPSK Modulation

\[
\begin{align*}
0 \rightarrow -1 \\
1 \rightarrow +1
\end{align*}
\]

\[
\begin{array}{c}
\text{Transmitted Constellation} \\
\text{Received Constellation}
\end{array}
\]

\[
x(t) \\
y(t) = x(t) + \nu(t)/h
\]

How well we can decode depends on SNR?

\[
\text{SNR} = \frac{\text{Signal Power}}{\text{Noise Power}} = \frac{E[|hx(t)|^2]}{E[|\nu(t)|^2]} = \frac{|h|^2E_s}{N'_0} = \frac{E_s}{N_0}
\]

Lump attenuation in noise
BPSK Modulation

Transmitted Constellation

Received Constellation

How well we can decode depends on SNR?

\[ SNR = \frac{Signal\ Power}{Noise\ Power} = \frac{E_s}{N_0} = 25\ dB \]
BPSK Modulation

$0 \rightarrow -1$
$1 \rightarrow +1$

Transmitted Constellation

Received Constellation

How well we can decode depends on SNR?

$SNR = \frac{Signal\ Power}{Noise\ Power} = \frac{E_s}{N_0} = 19\ dB$
**BPSK Modulation**

- $0 \rightarrow -1$
- $1 \rightarrow +1$

Transmitted Constellation

Received Constellation

How well we can decode depends on SNR?

$$SNR = \frac{Signal\ Power}{Noise\ Power} = \frac{E_s}{N_0} = 13\ dB$$
BPSK Modulation

Transmitted Constellation

\[ x(t) \]

Received Constellation

\[ y(t) = x(t) + v(t) \]

How well we can decode depends on SNR?

\[ SNR = \frac{\text{Signal Power}}{\text{Noise Power}} = \frac{E_s}{N_0} = 7 \text{ dB} \]
BPSK Modulation

How well we can decode depends on SNR?

\[ SNR = \frac{Signal\ Power}{Noise\ Power} = \frac{E_s}{N_0} = 3.5\ dB \]
Maximum Likelihood Decoder

\[ x = \pm 1 \quad \rightarrow \quad y = x + v \]

\[
P(x = +1 | y) \quad \quad 1 \quad \quad \quad \quad \quad \quad \quad P(x = -1 | y) \\
\begin{array}{c|c}
0 & \end{array}
\]

\[
P(x = +1) P(y | x = +1) \quad \quad 1 \quad \quad \quad \quad \quad \quad \quad P(x = -1) P(y | x = -1) \\
\begin{array}{c|c}
P(y) & 1 \\
0 & \end{array}
\]

\[ P(x = +1) = P(x = -1) = \frac{1}{2} \]

\[
P(y | x = +1) \quad \quad 1 \quad \quad \quad \quad \quad \quad \quad P(y | x = -1) \\
\begin{array}{c|c}
0 & \end{array}
\]
Maximum Likelihood Decoder

\[ x = \pm 1 \quad \rightarrow \quad y = x + v \]

\[ P(y|x = +1) \quad \frac{1}{\sqrt{2\pi\sigma^2}} \quad P(y|x = -1) \]

Guassian Noise: \( v \sim CN(0, \sigma) \quad \rightarrow \quad y|x \sim CN(x \sigma) \)

\[ P(v) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{|v|^2}{2\sigma^2}} \]

\[ 2\sigma^2 = N_0 \rightarrow \sigma = \sqrt{N_0/2} \]

\[ P(y|x = +1) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{|y-1|^2}{2\sigma^2}} \]

\[ P(y|x = -1) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{|y+1|^2}{2\sigma^2}} \]
Maximum Likelihood Decoder

\[ x = \pm 1 \quad \rightarrow \quad y = x + v \]

\[
P(y|x = +1) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{|y-1|^2}{2\sigma^2}}
\]

\[
P(y|x = -1) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{|y+1|^2}{2\sigma^2}}
\]

\[
P(y|x = +1) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{|y-1|^2}{2\sigma^2}}
\]

\[
P(y|x = -1) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{|y+1|^2}{2\sigma^2}}
\]

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\]

\[
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\]

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\]

\[
P(y|x = -1) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{|y+1|^2}{2\sigma^2}}
\]
Maximum Likelihood Decoder

\[ x = \pm 1 \quad \rightarrow \quad y = x + v \]

\[
P(y|x = +1) = \begin{cases} 1 & \text{if } y > 0 \\ 0 & \text{otherwise} \end{cases} \quad P(y|x = -1) = \begin{cases} 1 & \text{if } y < 0 \\ 0 & \text{otherwise} \end{cases}
\]

\[
-\frac{|y - 1|^2}{2\sigma^2} \quad \begin{cases} 1 & \text{if } y > 0 \\ 0 & \text{otherwise} \end{cases} \quad -\frac{|y + 1|^2}{2\sigma^2}
\]

\[
-|y - 1|^2 \quad \begin{cases} 1 & \text{if } y > 0 \\ 0 & \text{otherwise} \end{cases} \quad -|y + 1|^2
\]

\[
-(Re\{y\} - 1)^2 - Im\{y\}^2 \quad \begin{cases} 1 & \text{if } y > 0 \\ 0 & \text{otherwise} \end{cases} \quad -(Re\{y\} + 1)^2 - Im\{y\}^2
\]
Maximum Likelihood Decoder

\[
x = \pm 1 \quad \rightarrow \quad y = x + v
\]

\[
P(y|x = +1) \quad \frac{1}{\nabla} \quad \frac{1}{\nabla} \quad P(y|x = -1)
\]

\[
-(Re\{y\} - 1)^2 \quad \frac{1}{\nabla} \quad -(Re\{y\} + 1)^2
\]

\[
-Re\{y\}^2 + 2Re\{y\} - 1 \quad \frac{1}{\nabla} \quad -Re\{y\}^2 - 2Re\{y\} - 1
\]

\[
4Re\{y\} \quad \frac{1}{\nabla} \quad 0
\]
Maximum Likelihood Decoder

\[ x = \pm 1 \quad \rightarrow \quad y = x + v \]

\[ P(y|x = +1) \begin{cases} 1 & \text{if } y > 0 \\ \geq 0 & \text{else} \end{cases} \quad P(y|x = -1) \begin{cases} 1 & \text{if } y < 0 \\ \geq 0 & \text{else} \end{cases} \]

\[ Re\{y\} \begin{cases} 1 \quad \text{if } y > 0 \\ \geq 0 & \text{else} \end{cases} = 0 \]

\[ Q \]

\[ 0 \rightarrow -1 \]
\[ 1 \rightarrow +1 \]
4-QAM
Is the SNR the same in these 3 constellations?

\[
SNR = \frac{\text{Signal Power}}{\text{Noise Power}} = \frac{E[|x(t)|^2]}{E[|v(t)|^2]} = \frac{E_s}{N_0}
\]
16-QAM

4-QAM

BPSK

\[ SNR = \frac{E[|x(t)|^2]}{N_0} \]

\[ SNR = \frac{(4 \times 2 + 8 \times 10 + 4 \times 18)/16}{N_0} \]

\[ SNR = \frac{(2 + 2 + 2 + 2)/4}{N_0} \]

\[ SNR = \frac{(1 + 1)/2}{N_0} \]

\[ = \frac{10}{N_0} \] 

\[ > \]

\[ = \frac{2}{N_0} \] 

\[ > \]

\[ = \frac{1}{N_0} \]
16-QAM

4-QAM

BPSK

Transmit power is not the same!

Must not increase transmit power to use higher order modulation!
Normalize to maintain constant transmit power.

\[
\text{SNR}_{16-QAM} = \frac{(4 \times 2 + 8 \times 10 + 4 \times 18) / 16}{N_0} = \frac{10}{N_0}
\]

\[
\text{SNR}_{4-QAM} = \frac{(2 + 2 + 2 + 2) / 4}{N_0} = \frac{2}{N_0}
\]

\[
\text{SNR}_{BPSK} = \frac{(1 + 1) / 2}{N_0} = \frac{1}{N_0}
\]
Normalzize to maintain constant transmit power.

\[ SNR = \frac{(4 \times 2/10 + 8 \times 1 + 4 \times 18/10)}{N_0} / 16 \]
\[ SNR = \frac{(1 + 1 + 1 + 1)}{N_0} / 4 \]
\[ SNR = \frac{(1 + 1)}{2} / N_0 \]
16-QAM

4-QAM

BPSK

Normalize to maintain constant transmit power.

\[ SNR = \frac{1}{N_0} \]

\[ SNR = \frac{1}{N_0} \]

\[ SNR = \frac{1}{N_0} \]

Need Higher SNR to Decode Higher Order Modulation.
Higher Order Modulation:
• Needs higher SNR to decode correctly.
• Achieves higher bit rate

Given an SNR, choose highest order modulation that guarantees minimal Bit Error Rate (BER)
Bit Error Rate (BER) of BPSK

Encoding:
\[
\begin{align*}
    b = 0 & \rightarrow x = -1 \\
    b = 1 & \rightarrow x = +1
\end{align*}
\]

\[
y = x + v
\]

\[
BER = P(\hat{b} = 0 \cap b = 1) + P(\hat{b} = 1 \cap b = 0)
\]
\[
= P(b = 1)P(\hat{b} = 0|b = 1) + P(b = 0)P(\hat{b} = 1|b = 0)
\]
\[
= \frac{1}{2} P(\hat{b} = 0|b = 1) + \frac{1}{2} P(\hat{b} = 1|b = 0)
\]
\[
= \frac{1}{2} P(y < 0|b = 1) + \frac{1}{2} P(y > 0|b = 0)
\]
\[
= \frac{1}{2} P(y < 0|x = +1) + \frac{1}{2} P(y > 0|x = -1)
\]
Bit Error Rate (BER) of BPSK

Encoding:
\[
\begin{align*}
    b = 0 & \rightarrow x = -1 \\
    b = 1 & \rightarrow x = +1
\end{align*}
\]

\[
BER = \frac{1}{2} P(y < 0|x = +1) + \frac{1}{2} P(y > 0|x = -1)
\]

\[
= \frac{1}{2} \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}\sigma^2} e^{-\frac{(y-1)^2}{2\sigma^2}} \, dy + \frac{1}{2} \int_{0}^{\infty} \frac{1}{\sqrt{2\pi}\sigma^2} e^{-\frac{(y+1)^2}{2\sigma^2}} \, dy
\]

Decoding:
\[
\begin{align*}
    y = x + v \\
    P(y|x = +1) & \xrightarrow{\text{Re}\{y\}} \begin{cases} 1 & 0 \\
                                           0 & 1
\end{cases} \\
    P(y|x = -1) & \xrightarrow{\text{Re}\{y\}} \begin{cases} 0 & 1 \\
                                           0 & 1
\end{cases}
\end{align*}
\]

\[
Q = \frac{1}{\sqrt{2\pi}\sigma^2} e^{-\frac{(y+1)^2}{2\sigma^2}} \, dy
\]

\[
u = \frac{y + 1}{\sigma}
\]

\[
= \frac{1}{\sqrt{2\pi}} \int_{\frac{1}{\sigma}}^{\infty} e^{-\frac{u^2}{2}} \, du = \frac{1}{2} \text{erfc}\left(\frac{1}{\sigma\sqrt{2}}\right) = Q(1/\sigma) = Q(\sqrt{2}/N_0) = Q(\sqrt{2E_s/N_0}) = Q(\sqrt{2\text{SNR}})
\]

Error Function \hspace{2cm} Q Function
Bit Error Rate (BER) of BPSK

Encoding:
\[ b = 0 \rightarrow x = -1 \]
\[ b = 1 \rightarrow x = +1 \]

Decoding:
\[ y = x + v \]
\[ P(y|x = +1) \]
\[ P(y|x = -1) \]
\[ Re\{y\} \]
\[ Q \]

BER = \( Q(\sqrt{2SNR}) = Q(\sqrt{2E_s/N_0}) \)
Bit Error Rate (BER) of BPSK

Encoding:
\[ b = 0 \rightarrow x = -1 \]
\[ b = 1 \rightarrow x = +1 \]

Decoding:
\[ y = x + v \]
\[ P(y|x = +1) \begin{cases} 1 \\ 0 \end{cases} \quad P(y|x = -1) \begin{cases} 1 \\ 0 \end{cases} \]

\[ Re\{y\} \begin{cases} 1 \\ 0 \end{cases} \quad Q \begin{cases} 1 \\ 0 \end{cases} \]

\[ BER = Q(\sqrt{2SNR}) = Q(\sqrt{2E_s/N_0}) \]
Bit Error Rate (BER)

- Number of Constellation Points: $M$
- Bits per symbol: $\log_2 M$
- Signal-to-Noise Ratio: $SNR$
  
  - SNR per symbol: $\frac{E_s}{N_0}$
  
  - SNR per bit: $\frac{E_b}{N_0} = \frac{1}{\log_2 M} \frac{E_s}{N_0}$

- Symbol Error Rate: $SER = P_s$
- Bit Error Rate: $BER = P_b$

- Relation between $P_s$ and $P_b$?
  
  A. $P_b \leq P_s$  
  B. $P_b = P_s$  
  C. $P_b > P_s$
Bit Error Rate (BER)

- Number of Constellation Points: \( M \)
- Bits per symbol: \( \log_2 M \)
- Signal-to-Noise Ratio: \( SNR \)
  - SNR per symbol: \( \frac{E_s}{N_0} \)
  - SNR per bit: \( \frac{E_b}{N_0} = \frac{1}{\log_2 M} \frac{E_s}{N_0} \)
- Symbol Error Rate: \( SER = P_s \)
- Bit Error Rate: \( BER = P_b \)
- Relation between \( P_s \) and \( P_b \)?
  - A. \( P_b \leq P_s \)
  - B. \( P_b = P_s \)
  - C. \( P_b > P_s \)
Bit Error Rate (BER)

- Number of Constellation Points: $M$

- Bits per symbol: $\log_2 M$

- $P_b \leq P_s$?

1 symbol error can at most generate $\log_2 M$ bit errors.

$$P_b = \frac{\text{Number of Error Bits}}{\text{Total Number of Bits}} \leq \frac{\text{Number of Error Symbols} \times \log_2 M}{\text{Total Number of Symbols} \times \log_2 M} = P_s$$

How much is $P_b$ less than $P_s$?

Depends on the code used: How we assign bits to symbol
How much is $P_b$ less than $P_s$?

Depends on the code used: How we assign bits to symbol

$$P_b \approx \frac{3}{8} P_s$$

16-QAM

$$P_b \approx \frac{1}{4} P_s$$

Most likely error occurs between nearest neighbor constellation points!

Minimize bit flips between nearest neighbors.
How much is $P_b$ less than $P_s$?

Depends on the code used: How we assign bits to symbol

16-QAM

$P_b \approx \frac{1}{4} P_s$

Gray Codes

Bit flips between nearest neighbors $= 1$

$P_b \approx \frac{1}{\log_2 M} P_s$
Bit Error Rate (BER)

- Number of Constellation Points: $M$

- Bits per symbol: $\log_2 M$

- Assuming Gray Code is used:

  $$BER: \quad P_b = \frac{1}{\log_2 M} P_s$$

- Nearest Neighbor Approximation:

  $$SER: \quad P_s \approx \# \text{ nearest neighbors} \times Q\left(\frac{d_{min}}{\sqrt{2N_0}}\right)$$
Bit Error Rate (BER)

SER: \( P_s \approx \#nn \times Q \left( \frac{d_{\text{min}}}{\sqrt{2N_0}} \right) = Q \left( \frac{2\sqrt{E_s}}{\sqrt{2N_0}} \right) = Q \left( \frac{\sqrt{2E_s}}{N_0} \right) \)

BER: \( P_b = \frac{1}{\log_2 M} P_s = Q \left( \frac{\sqrt{2E_b}}{N_0} \right) \)

BPSK:

\[ d_{\text{min}} = 2 \sqrt{E_s} \]

\[ \#nn = 1 \]

\[ \log_2 M = 1 \]
Bit Error Rate (BER)

SER: $P_s \approx \#nn \times Q\left(\frac{d_{\text{min}}}{\sqrt{2N_0}}\right) = 2Q\left(\sqrt{E_s/N_0}\right)$

BER: $P_b = \frac{1}{\log_2 M} P_s = Q\left(\sqrt{2E_b/N_0}\right)$

4-QAM/QPSK

$$d_{\text{min}} = \frac{2}{\sqrt{2}} \sqrt{E_s}$$

$$\#nn = 2$$

$$\log_2 M = 2$$
Bit Error Rate (BER)

**SER:** \( P_s \approx \#nn \times Q \left( \frac{d_{\text{min}}}{\sqrt{2N_0}} \right) = 2Q\left(\sqrt{\frac{E_s}{N_0}}\right) \)

**BER:** \( P_b = \frac{1}{\log_2 M} P_s = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) \)

---

**4-QAM/QPSK**

- \( d_{\text{min}} = \frac{2}{\sqrt{2}} \sqrt{E_s} \)
- \( \#nn = 2 \)
- \( \log_2 M = 2 \)
Bit Error Rate (BER)

**SER:** \( P_s \approx \#nn \times Q \left( \frac{d_{\text{min}}}{\sqrt{2N_0}} \right) = 4Q(\sqrt{E_s/5N_0}) \)

**BER:** \( P_b = \frac{1}{\log_2 M} P_s = Q(\sqrt{4E_b/5N_0}) \)

16-QAM

\[ d_{\text{min}} = \frac{2}{\sqrt{10}} \sqrt{E_s} \]
\[ \#nn = 4 \]
\[ \log_2 M = 4 \]
Bit Error Rate (BER)

SER: \( P_s \approx \#nn \times Q\left( \frac{d_{\text{min}}}{\sqrt{2N_0}} \right) = 4Q\left( \sqrt{\frac{3E_s/N_0}{M - 1}} \right) \)

BER: \( P_b = \frac{1}{\log_2 M} P_s = \frac{4}{\log_2 M} Q\left( \sqrt{\frac{3\log_2 M E_b/N_0}{M - 1}} \right) \)

M-QAM

\[ d_{\text{min}} = \sqrt{\frac{6E_s}{M - 1}} \]

\[ \#nn = 4 \]

\[ \log_2 M \]
Bit Error Rate (BER)

\[ \text{SER: } P_s \approx #nn \times Q \left( \frac{d_{\text{min}}}{\sqrt{2N_0}} \right) = 2Q \left( \sqrt{2E_s/N_0 \sin \left( \frac{\pi}{M} \right)} \right) \]

\[ \text{BER: } P_b = \frac{1}{\log_2 M} P_s = \frac{2}{\log_2 M} Q \left( \sqrt{2 \log_2 M E_b/N_0 \sin \left( \frac{\pi}{M} \right)} \right) \]

M-PSK

\[ d_{\text{min}} = 2 \sin \left( \frac{\pi}{M} \right) \sqrt{E_s} \]

\[ #nn = 2 \]

\[ \log_2 M \]
# Bit Error Rate (BER)

**SER:** \( P_s \approx \#mn \times Q\left(\frac{d_{\text{min}}}{\sqrt{2N_0}}\right) \)

**BER:** \( P_b = \frac{1}{\log_2 M} P_s \)

<table>
<thead>
<tr>
<th>Modulation Scheme</th>
<th>( Q(\sqrt{2E_s/N_0}) = \frac{2}{\pi} )</th>
<th>( Q(\sqrt{2E_b/N_0}) = \frac{2}{\pi} )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>BPSK</strong></td>
<td>( Q(\sqrt{2E_s/N_0}) )</td>
<td>( Q(\sqrt{2E_b/N_0}) )</td>
</tr>
<tr>
<td><strong>QPSK/4-QAM</strong></td>
<td>( 2Q(\sqrt{E_s/N_0}) )</td>
<td>( 2Q(\sqrt{E_b/N_0}) )</td>
</tr>
<tr>
<td><strong>MPAM</strong></td>
<td>( \frac{2(M-1)}{M} Q\left(\frac{6E_s/N_0}{M^2 - 1}\right) )</td>
<td>( \frac{2(M-1)}{M \log_2 M} Q\left(\frac{6 \log_2 M E_b/N_0}{M^2 - 1}\right) )</td>
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</tr>
<tr>
<td><strong>MQAM</strong></td>
<td>( 4Q\left(\sqrt{\frac{3E_s/N_0}{M-1}}\right) )</td>
<td>( \frac{4}{\log_2 M} Q\left(\sqrt{\frac{3 \log_2 M E_b/N_0}{M-1}}\right) )</td>
</tr>
</tbody>
</table>

\( \alpha_M Q\left(\sqrt{\beta_M E_s/N_0}\right) \quad \hat{\alpha}_M Q\left(\sqrt{\hat{\beta}_M E_b/N_0}\right) \)

\( \hat{\alpha}_M = \frac{\alpha_M}{\log_2 M} \), \( \hat{\beta}_M = \beta_M \log_2 M \), \( \frac{E_b}{N_0} = \frac{1}{\log_2 M} \frac{E_s}{N_0} \)
Bit Error Rate (BER)

BER

$E_b/N_0$ in dB

- BPSK/4QAM
- 16 QAM
- 64 QAM
Bit Error Rate (BER)

Approximate BER for different modulation schemes:

- BPSK/4QAM: \( \approx 7 \text{ dB} \)
- 16 QAM: \( \approx 15 \text{ dB} \)
- 64 QAM:
Shape of the Constellation (e.g. $M = 16$)

16-QAM:
4×4 Grid

16-PSK:
Circular

2×8 Grid

Random
Shape of the Constellation (e.g. $M = 16$)

SER: $P_s \approx #nn \times Q \left( \frac{d_{\text{min}}}{\sqrt{2N_0}} \right)$

Goal: Maximize $d_{\text{min}}$ for the same overall power

$4 \times 4$ Grid

$2 \times 8$ Grid

$$d_{\text{min}} = \frac{2}{\sqrt{10}} \sqrt{E_s}$$

$$d_{\text{min}} = \frac{2}{\sqrt{352}} \sqrt{E_s}$$
Shape of the Constellation (e.g. $M = 16$)

$SER: P_s \approx \#nn \times Q \left( \frac{d_{min}}{\sqrt{2N_0}} \right)$

Goal: Maximize $d_{min}$ for the same overall power

Average of constellation should be zero.

$E[|x(t)|^2] = 1$

If Average is not zero, we have higher TX Power for the same $d_{min}$ or smaller $d_{min}$ for same TX power.
Shape of the Constellation (e.g. $M = 16$)

$SER: P_s \approx \#nn \times Q \left( \frac{d_{\text{min}}}{\sqrt{2N_0}} \right)$

Goal: Maximize $d_{\text{min}}$ for the same overall power

Average of constellation should be zero.

Geometric packing problem.
Shape of the Constellation (e.g. $M = 16$)

SER: $P_s \approx \#nn \times Q \left( \frac{d_{\text{min}}}{\sqrt{2N_0}} \right)$

Goal: Maximize $d_{\text{min}}$ for the same overall power

Average of constellation should be zero.

Geometric packing problem.

Rotating the constellation maintains same $d_{\text{min}}$ for same overall power
Shape of the Constellation (e.g. $M = 16$)

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QAM Is Popular!

- **WiFi**
  - 802.11n: BPSK, QPSK, 16-QAM, 64-QAM
  - 802.11ac: BPSK, QPSK, 16-QAM, 64-QAM, 256-QAM
  - 802.11ax (2019): BPSK, QPSK, 16-QAM, 64-QAM, 256-QAM, 1024-QAM

- **LTE**: QPSK, 16-QAM, 64-QAM

- **Digital TV**:
  - DVB-C: 16-QAM, 64-QAM, 256-QAM
  - DVB-C2: 16-QAM, 64-QAM, 256-QAM, 1024-QAM, 4096-QAM
  - Next Gen. to include 16364-QAM & 65536-QAM

- **Ethernet, Phone lines, Power lines...**: 1024-QAM, 4096-QAM

- **ADSL**: 32768-QAM
65536-QAM

- Number of Constellation Points: $M = 65536$
- Bits per symbol: $\log_2 M = 16$
- QAM Grid: 256×256
- Need 8 bits to represent $I$ and 8 bits to represent $Q$

What is the problem?

ADCs need to have a large dynamic range!

Quantization noise kicks in!
65536-QAM

- Number of Constellation Points: $M = 65536$
- Bits per symbol: $\log_2 M = 16$
- QAM Grid: $256 \times 256$
- Need 8 bits to represent $I$ and 8 bits to represent $Q$
- ADC with $K$ bits typically supports $N < K$ bit samples

ENOdB: Effective Number of Bits $N < K$
Quantization Noise

• Consider N bit quantization

\[ SNR_{\text{quant.}}(\text{dB}) = 10 \log_{10} 2^{2N} = 20N \log_{10} 2 = 6.02N \]

\[ \approx 6 \text{ dB per bit} \]
Quantization Noise

- Consider N bit quantization

\[ SNR_{quant.} = \frac{(Signal \ Amplitude)^2}{(Quantization \ Noise)^2} \]

\[ = \frac{(2^{N-1})^2}{(0.5)^2} = \frac{2^{2N-2}}{2^{-2}} = 2^{2N} \]

\[ SNR_{quant.}(dB) = 10 \log_{10} 2^{2N} = 20N \log_{10} 2 \approx 6.02N \]

\approx 6 \text{ dB per bit}

Need: \( SNR_{quant.} > SNR_{thermal} \)

65536-QAM: \( BER = \frac{1}{4} Q \left( \sqrt{\frac{3 \times SNR}{65535}} \right) \)

Need: \( SNR > 52 \text{ dB} \) to achieve \( BER < 10^{-3} \)

\[ \Rightarrow \text{Need: } N > 9 \text{ bits} \]
65536-QAM

- Number of Constellation Points: $M = 65536$
- Bits per symbol: $\log_2 M = 16$
- QAM Grid: $256 \times 256$
- Need 8 bits to represent $I$ and 8 bits to represent $Q$
- ADC with $K$ bits typically supports $N < K$ bit samples

**ENOB: Effective Number of Bits $N < K$**

- Need very high SNR to achieve reasonable BER.
- Need to minimize quantization noise: $N \geq 12$

Need at least a 14 bit or 16 bit ADC
Quantization Noise

• Consider N bit quantization

SNR\text{quant.} = \frac{(\text{Signal Amplitude})^2}{(\text{Quantization Noise})^2}

= \frac{(2^{N-1})^2}{(0.5)^2} = \frac{2^{2N-2}}{2^{-2}} = 2^{2N}

SNR\text{quant.}(\text{dB}) = 10 \log_{10} 2^{2N} = 20N \log_{10} 2 \approx 6.02N

≈ 6 \text{ dB per bit}

Assumes signal amplitude fills maximum quantization level.

Signals arrives too attenuated → fill smaller quantization levels → Low \text{SNR}_{\text{quant.}}
Quantization Noise

• Consider N bit quantization

\[ SNR_{\text{quant.}} = \frac{(\text{Signal Amplitude})^2}{(\text{Quantization Noise})^2} \]

\[ = \frac{(2^{N-1})^2}{(0.5)^2} = \frac{2^{2N-2}}{2^{-2}} = 2^{2N} \]

\[ SNR_{\text{quant.}}(\text{dB}) = 10 \log_{10} 2^{2N} = 20N \log_{10} 2 = 6.02N \approx 6 \text{ dB per bit} \]

Assumes signal amplitude fills maximum quantization level.

Signals arrives too attenuated \(\Rightarrow\) fill smaller quantization levels \(\Rightarrow\) Low \(SNR_{\text{quant.}}\).

Signals arrives too amplified \(\Rightarrow\) Clipping
AGC: Automatic Gain Control

- Consists of Variable Gain Amplifier and Control Circuit
- Adjust the gain to minimize quantization noise & avoid clipping.
- Receiver Circuit:
Definitions & Variables

- \( x(t) \): Transmitted Signal
- \( v(t) \): Additive Gaussian Noise
- \( y(t) \): Received Signal
- \( h \): Single Tap Channel Coefficient.
- \( N'_0 \): Gaussian Noise Energy
- \( N_0 \): Gaussian Noise Scaled by Channel
- \( E[ \cdot ] \): Expectation
- \( SNR \): Signal to Noise Ratio
- \( E_s \): Energy per Symbol
- \( E_b \): Energy per Bit
- \( P(\cdot) \): Probability Distribution
- \( P(\cdot|\cdot) \): Conditional Probability Distribution
- \( \sigma \): Std. Dev. of Gaussian Noise.

- \( b \): Transmitted Bit
- \( \hat{b} \): Decoded Bit
- \( BER \): Bit Error Rate
- \( SER \): Symbol Error Rate
- \( P_B \): Probability of Bit Error
- \( P_S \): Probability of Symbol Error
- \( Q(\cdot) \): Q Function
- \( M \): Number of Constellation Points
- \( d_{min} \): Minimum Distance between constellation points.
- \#nn\): Number of nearest neighbors in the constellation.
- \( K \): Number of ADC bits
- \( N \): Effective Number of ADC bits
- \( \alpha_M, \beta_M, \hat{\alpha}_M, \hat{\beta}_M \): Parameters of BER Q function