ECE 463: Digital Communications Lab.

Lecture 6: Carrier Frequency Offset
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Previous Lecture:

✓ Multipath Channel
✓ Channel Estimation & Correction
✓ Narrowband vs. Wideband Channels
✓ Channel Equalization

This Lecture:

❑ Carrier Frequency Offset Estimation & Correction
❑ Carrier Recovery & Phase Tracking
Digital Communication System

1011010110011001

TX

RX

1011010110011001

Bits

Bits-to-Symbols Mapper

Modulation (Encoding)

Pulse Shaping

DAC

LPF

Mixer

BPF

PA

PLL

Demodulation (Decoding)

Symbols-to-Bits Mapper

Channel Equalization

Frame Sync.

Matched Filter

Symbol Timing Recovery

Sync.

ADC

LPF

BPF

LNA

PPA

Mixer
Digital Communication System

So far, we assumed carriers generated by LOs are perfectly synchronized!
So far, we assumed carriers generated by PLLs are perfectly synchronized!
Carrier Frequency Offset

\[ x(t) \rightarrow x(t) \times e^{-j2\pi f_c t} \rightarrow \alpha x(t - \tau) e^{-j2\pi f_c (t-\tau)} \]
Carrier Frequency Offset

\[ x(t) \rightarrow x(t) e^{-j2\pi f_c t} \rightarrow \alpha e^{j2\pi f_c \tau} x(t - \tau) e^{-j2\pi f_c t} \]
Carrier Frequency Offset

\[ x(t) \rightarrow x(t) e^{-j2\pi f_c t} \rightarrow h x(t - \tau) e^{-j2\pi f_c t} \rightarrow h x(t - \tau) e^{-j2\pi f_c t} \times e^{j2\pi f_c t} \]

\[ \rightarrow h x(t - \tau) \]

\[ \rightarrow y(t) = h x(t - \tau) + v(t) \]

Assumes TX & RX perfectly synched
Carrier Frequency Offset

\[ x(t) \rightarrow x(t) \times e^{-j2\pi f_c t} \rightarrow h x(t - \tau) e^{-j2\pi f_c t} \rightarrow h x(t - \tau) e^{-j2\pi f_c t} \times e^{j2\pi f'_c t} \]

TX & RX are not synched

\[ y(t) = h x(t - \tau) e^{-j2\pi \Delta f_c t} + v(t) \]

Phase changes with time!
Carrier Frequency Offset

Consider BPSK Modulation.

\[ 0 \rightarrow -1 \]
\[ 1 \rightarrow +1 \]

\[ x(t) \]

\[ h x(t - \tau) e^{-j2\pi f_c t} + v(t) \]
Consider BPSK Modulation.

\[
\begin{align*}
0 & \rightarrow -1 \\
1 & \rightarrow +1
\end{align*}
\]

\[
x(t) \quad \text{and} \quad h x(t - \tau) e^{-j2\pi f_c t} + v(t)
\]

Impossible to Decode!
Carrier Frequency Offset

Consider 16 QAM Modulation

Need to estimate and correct CFO to decode!
Consequences of CFO

• Cannot decode bits correctly

• Correlation with training sequence does not work for frame synchronization.
Frame Synchronization without CFO

- $t[n]$ is training sequence of length $N$

- Discrete samples: $y[n] = h \cdot t[n - d] + v[n]$

$$R[n] = \frac{1}{N} \left| \sum_{k=0}^{N-1} t^*[k] y[n + k] \right|^2$$

$$= \frac{1}{N} \left| \sum_{k=0}^{N-1} t^*[k] h \cdot t[n + k - d] + \sum_{k=0}^{N-1} t^*[k] v[n + k] \right|^2$$
Frame Synchronization without CFO

- $t[n]$ is training sequence of length $N$

- Discrete samples: $y[n] = h \cdot t[n - d] + v[n]$

\[
R[n] = \frac{1}{N} \left| \sum_{k=0}^{N-1} t^*[k]y[n + k] \right|^2
\]

\[
= \frac{1}{N} \left| \sum_{k=0}^{N-1} t^*[k]h \cdot t[n + k - d] + \sum_{k=0}^{N-1} t^*[k]v[n + k] \right|^2
\]
Frame Synchronization without CFO

-1 + 1 − 1 − 1 + 1 \ldots − 1 + 1 − 1 − 1 + 1 + 1 + 1 − 1 \ldots

- $t[n]$ is training sequence of length $N$

- Discrete samples: $y[n] = h \cdot t[n - d] + v[n]$

$$R[n] = \frac{1}{N} \left| \sum_{k=0}^{N-1} t^*[k] y[n + k] \right|^2$$

$$= \frac{1}{N} \left| \sum_{k=0}^{N-1} t^*[k] h \cdot t[n + k - d] \right|^2 = |h|^2 \begin{cases} 1 \quad \text{if} \quad n = d \\ 1/N \quad \text{if} \quad n \neq d \end{cases}$$
Frame Synchronization with CFO

- $t[n]$ is training sequence of length $N$

- Discrete samples: $y[n] = h \cdot t[n - d]e^{-j2\pi \Delta f_c n T_s} + v[n]$

$$R[n] = \frac{1}{N} \left| \sum_{k=0}^{N-1} t^*[k] \times h \cdot t[n + k] e^{-j2\pi \Delta f_c (n+k) T_s} + \sum_{k=0}^{N-1} t^*[k]v[k] \right|^2$$
Frame Synchronization with CFO

-1 + 1 − 1 − 1 + 1 ... − 1 + 1 − 1 − 1 + 1 + 1 + 1 − 1 ...

- $t[n]$ is training sequence of length $N$

- Discrete samples: $y[n] = h \cdot t[n − d]e^{−j2πf_c nT_s} + v[n]$

$$R[n] = \frac{1}{N} \left| \sum_{k=0}^{N−1} t^*[k] × h \cdot t[n + k]e^{−j2πf_c (n+k)T_s} + \sum_{k=0}^{N−1} t^*[k]v[k] \right|^2$$
Frame Synchronization with CFO

\[-1 + 1 - 1 - 1 + 1 \quad \cdots - 1 + 1 - 1 - 1 + 1 + 1 + 1 - 1 \cdots\]

- \( t[n] \) is training sequence of length \( N \)

- Discrete samples: \( y[n] = h \cdot t[n - d] e^{-j2\pi\Delta f_c n T_s} + v[n] \)

\[
R[n] = \frac{1}{N} |h|^2 \left| \sum_{k=0}^{N-1} t^*[k] t[n + k - d] e^{-j2\pi\Delta f_c (n+k) T_s} \right|^2
\]

\[
= \frac{|h|^2}{N} \begin{cases} 
\left| \sum_{k=0}^{N-1} e^{-j2\pi\Delta f_c (n+k) T_s} \right|^2 & \text{if } n = d \\
1 & \text{if } n \neq d
\end{cases}
\]
Frame Synchronization with CFO

-1 + 1 − 1 − 1 + 1 ... − 1 + 1 − 1 − 1 + 1 + 1 + 1 − 1 ...

- $t[n]$ is training sequence of length $N$

- Discrete samples: $y[n] = h \cdot t[n - d]e^{-j2\pi f_c nT_s} + v[n]$

$$R[n] = \frac{1}{N} |h|^2 \left| \sum_{k=0}^{N-1} t^*[k]t[n + k - d]e^{-j2\pi f_c (n+k)T_s} \right|^2$$

$$= \frac{|h|^2}{N} \left\{ \left| \sum_{k=0}^{N-1} e^{-j2\pi f_c (n+k)T_s} \right|^2 \right\}$$

0 if $n = d$

$\approx$ noise

1 if $n \neq d$
Frame Synchronization with CFO

-1 + 1 - 1 - 1 + 1 -1 + 1 - 1 - 1 + 1 \cdots - 1 + 1 - 1 + 1 + 1 + 1 + 1 - 1 \cdots

- $t[n]$ is training sequence of length $N$

- Discrete samples: $y[n] = h \cdot t[n - d] e^{-j2\pi \Delta f_c n T_s} + v[n]$

- Repeat training sequence: $y[n + N] = y[n] = t[n]$

- Correlate first $N$ samples with next $N$ samples.

\[
R[n] = \left| \sum_{k=0}^{N-1} y^*[n + k] y[n + k + N] \right|
\]

if $n = d$, \[
= \left| \sum_{k=0}^{N-1} h^* t^*[k] e^{j2\pi \Delta f_c (n+k) T_s} \cdot h t[k] e^{-j2\pi \Delta f_c (n+N+k) T_s} \right|
\]

\[
= \left| h^2 e^{-j2\pi \Delta f_c N T_s} \sum_{k=0}^{N-1} t^*[k] t[k] \right| = N |h|^2
\]
Frame Synchronization with CFO

- $t[n]$ is training sequence of length $N$
- Discrete samples: $y[n] = h \cdot t[n - d] e^{-j2\pi f_c n T_s} + v[n]$
- Repeat training sequence: $y[n + N] = y[n] = t[n]$
- Correlate first $N$ samples with next $N$ samples.

$$R[n] = \left| \sum_{k=0}^{N-1} y^*[n + k] y[n + k + N] \right|$$

if $n = d$, $R[n] = N|h|^2$

if $n \neq d$, $R[n] < R[d]$

Find $n$ that maximizes $R[n]$. 
Frame Synchronization with CFO

-1 + 1 − 1 − 1 + 1 − 1 + 1 − 1 + 1 + 1 + 1 + 1 − 1 ... 

- **t[n]** is training sequence of length \( N \)

- Discrete samples: 
  \[ y[n] = h \cdot t[n - d]e^{-j2\pi f_c n T_s} + v[n] \]

- Repeat training sequence: 
  \[ y[n + N] = y[n] = t[n] \]

- Correlate first \( N \) samples with next \( N \) samples.

\[
R[n] = \frac{|\sum_{k=0}^{N-1} y^*[n+k]y[n+k+N]|}{\sqrt{\sum_{k=0}^{N-1}|y[n+k]|^2} \sqrt{\sum_{k=0}^{N-1}|y[n+k+N]|^2}} \begin{cases} 
= 1 & \text{if} \quad n = d \\
< 1 & \text{if} \quad n \neq d
\end{cases}
\]
Consequences of CFO

- Cannot decode bits correctly

Estimate & Correct for CFO

✓ Correlation with training sequence does not work for frame synchronization.

Repeat training sequence & correlate consecutive sequences of received samples.
Estimating & Correcting for CFO

- Use training sequence $t[n]$ of length $N$ repeated twice

- Discrete samples: $y[n] = h \cdot t[n - d]e^{-j2\pi\Delta f_c n T_s} + v[n]$

\[
A = \sum_{k=0}^{N-1} y^*[d + k]y[d + k + N]
\]

\[
= \sum_{k=0}^{N-1} h^* t^*[k]e^{j2\pi\Delta f_c (d+k) T_s} \cdot h t[k]e^{-j2\pi\Delta f_c (d+N+k) T_s} = N|h|^2 e^{-j2\pi\Delta f_c NT_s}
\]

Estimate CFO: $\Delta f_c = \frac{\angle A}{2\pi NT_s}$

Correct CFO: $y[n] \times e^{j2\pi\Delta f_c n T_s}$
Estimating & Correcting for CFO

Estimate CFO: \( \Delta f_c = \frac{\angle A}{2\pi NT_s} \)

Correct CFO: \( y[n] \times e^{j2\pi f_c nT_s} \)

Phase Wraps Around 2\(\pi\)

\(-\pi \leq \angle A \leq \pi\)
Estimating & Correcting for CFO

Estimate CFO: \[ \Delta f_c = \frac{\angle A}{2\pi NT_s} \]

Correct CFO: \[ y[n] \times e^{j2\pi f_c n T_s} \]

Phase Wraps Around 2\pi

\[ |\angle A| \leq \pi \]

\[ \Delta f_c \leq \frac{1}{2NT_s} \]

- \( N \) must not be too large to correctly estimate CFO

\[ \text{e.g., } f_c = 5 \text{ GHz, Clock Precision: 20 ppm, Bandwidth} = 10 \text{ MHz} \]

\[ \Delta f_c = 100 \text{ KHz, } T_s = 0.1 \mu s \]

\[ N \leq 50 \]
Estimating & Correcting for CFO

Estimate CFO: \( \Delta f_c = \frac{\angle A}{2\pi NT_s} \)

Correct CFO: \( y[n] \times e^{j2\pi \Delta f_c nT_s} \)

Phase Wraps Around 2\( \pi \)

\( |\angle A| \leq \pi \)

\( \Delta f_c \leq \frac{1}{2NT_s} \)

- \( N \) must not be too large to correctly estimate CFO
- \( N \) must be large enough to average out the noise
Estimating & Correcting for CFO

Estimate CFO: \[ \Delta f_c = \frac{\angle A}{2\pi NT_s} \]

Correct CFO: \[ y[n] \times e^{j2\pi \Delta f_c nT_s} \]

Phase Wraps Around 2\pi

\[ |\angle A| \leq \pi \]

\[ \Delta f_c \leq \frac{1}{2NT_s} \]

- \( N \) must not be too large to correctly estimate CFO
- \( N \) must be large enough to average out the noise

How do we know exact index value of \( n \)?

\[ y[n] \times e^{j2\pi \Delta f_c (n+k)T_s} = hx[n]e^{-j2\pi \Delta f_c nT_s} \times e^{j2\pi \Delta f_c (n+k)T_s} \]

\[ = hx[n]e^{j2\pi \Delta f_c kT_s} \]

\[ = (he^{j2\pi \Delta f_c kT_s})x[n] \]

\[ = h'x[n] \]

- It does not matter!
- Lumped with Channel Equalization
Consequences of CFO

- Cannot decode bits correctly

Estimate & Correct for CFO

- Correlation with training sequence does not work for frame synchronization.

  Repeat training sequence & correlate consecutive sequences of received samples.
Estimate & Correct for CFO

So Far: Pilot Assisted Carrier Acquisition

- Residual CFO:

\[ y(t) = h x(t - \tau)e^{-j2\pi \Delta f_c t} + \nu(t) \]

\[ \Delta f_c = d f_c + \delta f_c \]

Coarse CFO  Residual CFO

We estimated and corrected for coarse CFO!

Even small residual can accumulate over time to create large phase: \( e^{-j2\pi \delta f_c t} \)

Need to track the phase
Estimate & Correct for CFO

So Far: Pilot Assisted Carrier Acquisition

• Residual CFO: $\delta f_c$

• Initial Carrier Phase:

$$y(t) = h x(t - \tau) e^{-j(2\pi f_c t + \phi)} \times e^{j(2\pi f_c' t + \theta)}$$

$$y(t) = h x(t - \tau) e^{-j(2\pi f_c t + \phi - \theta)}$$

In principle, not a problem!
Lump it with Channel Equalization.

However, phase might not be stable over time.

Need to track the phase
Estimate & Correct for CFO

So Far: Pilot Assisted Carrier Acquisition

• Residual CFO: $\delta f_c$

• Initial Carrier Phase: $\phi$

Phase Tracking

Extract the phase of the carrier and track it over time.
Phase Tracking

Many Methods:

• Squared Difference Loop

• Phased Locked Loop

• Costas Loop

• ...

...
Squared Difference Loop Method

Consider simple BPSK: \( r(t) = \pm 1 \cos(2\pi f_0 t + \phi) \)

- Square the signal: \( r^2(t) = \cos^2(2\pi f_0 t + \phi) \)
- Bandpass filter it at \( 2f_0 \): \( r_p(t) = \cos(4\pi f_0 t + 2\phi) \)
- Sample it: \( r_p(kT_s) = \cos(4\pi f_0 kT_s + 2\phi) \)

Goal is to find & track \( \phi \)

Find \( \theta \) that minimizes the average squared error:

\[
J_{SD}(\theta) = \frac{1}{4} \text{LPF}\{ (r_p(kT_s) - \cos(4\pi f_0 kT_s + 2\theta))^2 \}
\]

\[
= \frac{1}{4} (1 - \cos(2\phi - 2\theta))
\]
Squared Difference Loop Method

Find $\theta$ that minimizes the average squared error:

$$J_{SD}(\theta) = \frac{1}{4} \text{LPF} \left\{ (r_p(kT_s) - \cos(4\pi f_0 kT_s + 2\theta))^2 \right\}$$

Minimize using Gradient Descent:

$$\theta[k + 1] = \theta[k] - \mu \frac{dJ_{SD}(\theta)}{d\theta} \bigg|_{\theta=\theta[k]}$$

$$\theta[k + 1] = \theta[k] - \mu \text{LPF} \left\{ (r_p(kT_s) - \cos(4\pi f_0 kT_s + 2\theta[k])) \sin(4\pi f_0 kT_s + 2\theta[k]) \right\}$$

$$\theta[k + 1] = \theta[k] - \mu \sin(2\phi - 2\theta[k])$$
Phased Lock Loop Method

Find $\theta$ that maximizes correlation:

$$J_{PLL}(\theta) = \frac{1}{2} \text{LPF}\{r_p(kT_s) \cos(4\pi f_0 kT_s + 2\theta)\}$$

Minimize using Gradient Descent:

$$\theta[k + 1] = \theta[k] - \mu \frac{dJ_{PLL}(\theta)}{d\theta}\bigg|_{\theta=\theta[k]}$$

$$\theta[k + 1] = \theta[k] - \mu \text{LPF}\{r_p(kT_s) \sin(4\pi f_0 kT_s + 2\theta[k])\}$$
Costas Loop Method

Find $\theta$ that maximizes correlation but operates directly on received signal:

$$J_c(\theta) = \frac{1}{2} \text{LPF}((\text{LPF}\{r(kT_s) \cos(2\pi f_0 kT_s + \theta)\}))^2$$

Minimize using Gradient Descent:

$$\theta[k + 1] = \theta[k] - \mu \frac{dJ_c(\theta)}{d\theta} \bigg|_{\theta=\theta[k]}$$

$$\theta[k + 1] = \theta[k] - \mu \text{LPF}\{(r(kT_s) \sin(2\pi f_0 kT_s + \theta[k]))\} \times \text{LPF}\{(r(kT_s) \sin(2\pi f_0 kT_s + \theta[k]))\}$$
Phase Tracking

Similar objective functions

\[ J_{SD}(\theta) \]

\[ J_{PLL}(\theta) \]

\[ J_C(\theta) \]
Definitions & Variables

- \( x(t) \): Transmitted Signal
- \( \nu(t) \): Additive Gaussian Noise
- \( y(t) \): Received Signal
- \( \tau \): Time delay of the signal
- \( h \): Single Tap Channel Coefficient.
- \( \alpha \): Attenuation of the channel
- \( f_c \): Carrier Frequency of Transmitter
- \( f_c' \): Carrier Frequency of Receiver
- \( \Delta f_c \): Carrier Frequency Offset (CFO)
- \( df_c \): Carrier Frequency Offset
- \( \delta f_c \): Carrier Frequency Offset
- \( \phi \): Initial TX Carrier Phase
- \( \theta \): Initial RX Carrier Phase
- \((\quad)^*\): Complex Conjugate
- \( T_s \): Symbol time
- \( y[n] \): Sampled received signal
- \( t[n] \): Training Sequence
- \( N \): Length of the Training Sequence
- \( R[n] \): Cross Correlation Function
- \( d \): Signal delay in number of samples
- \( n \): Symbol index
- \( r(t) \): Modulated Signal
- \( f_0 \): Intermediate Carrier Frequency
- \( r_p(t) \): Squared and bandpass filtered \( r(t) \)
- \( J_{SD}(\theta) \): Squared Difference Loop Optimization Function
- \( J_{PLL}(\theta) \): Phase Lock Loop Optimization Function
- \( J_C(\theta) \): Costas Loop Optimization Function
- \( \mu \): Gradient Descent update parameter