Wireless Communication

Channel Model for Mobile Communications

So far we studied the complex baseband model for point-to-point communications shown in Figure 1. Our goal now is to modify this channel model to incorporate the effects of the mobility. We will focus on terrestrial mobile communications channels – satellite channels are more “well-behaved”. The following are points worth noting in making the transition to the mobile communication channel model.

○ The additive noise term $w(t)$ is always present whether the channel is point-to-point or mobile, and usually $w(t)$ is modelled as proper complex WGN.

○ For point-to-point communications the channel response is generally well modelled by a linear time invariant (LTI) system ($h(\xi)$ may or may not be known at the receiver). For mobile communications, the channel response is time-varying, and we will see that it is well modelled as a linear time-varying (LTV) system.

To study the mobile communications channel, consider the situation where the mobile station (MS) is at location $(x, y)$ or $(d, \varphi)$ in a coordinate system with the base station (BS) at the origin as shown in Figure 2. A 3-d model may be more appropriate in some situations, but for simplicity we will consider a 2-d model. Also, we restrict our attention now to the channel connecting one pair of transmit (Tx) and receive (Rx) antennas.

If the mobile is fixed at location $(d, \varphi)$, the channel that it sees is time-invariant. The response of this time-invariant channel is a function of the location, and is determined by all paths connecting the BS and the MS. Thus we have the system shown in Figure 3, where $h_{d, \varphi}(\xi)$ is the impulse response of a causal LTI system, which is a function of the multipath profile between the BS and MS.

Referring to Figure 2, suppose the $n$-th path connecting the BS and MS has amplitude gain $\beta_n(d, \varphi)$ and delay $\tau_n(d, \varphi)$. The delay of $\tau_n(d, \varphi)$ introduces a carrier phase shift of

$$\phi_n(d, \varphi) = -2\pi f_c \tau_n(d, \varphi) + \text{constant}$$

where the constant depends on the reflectivity of the surface(s) that reflect the path. Then we can
write the output $y(t)$ in terms of the input $s(t)$ as

$$y(t) = \sum_n \beta_n(d, \varphi) e^{j\phi_n(d, \varphi)} s(t - \tau_n(d, \varphi))$$

which implies that the impulse response is

$$h_{d, \varphi}(\xi) = \sum_n \beta_n(d, \varphi) e^{j\phi_n(d, \varphi)} \delta(\xi - \tau_n(d, \varphi))$$

**Scales of Variation**

As the MS moves, $(d, \varphi)$ change with time and the linear system associated with the channel becomes LTV. There are two scales of variation:

- The first is a small-scale variation due to rapid changes in the phase $\phi_n$ as the mobile moves over distances of the order of a wavelength of the carrier $\lambda_c = c/f_c$, where $c$ is the velocity of light. This is because movements in space of the order of a wavelength cause changes in $\tau_n$ of the order of $1/f_c$, which in turn cause changes in $\phi_n$ of the order of $2\pi$. (Note that for a 900 MHz carrier, $\lambda_c \approx 1/3$ m.)

Modeling the phases $\phi_n$ as independent Uniform$[0, 2\pi]$ random variables, we can see that the average power gain in the vicinity of $(d, \varphi)$ is given by $\sum_n \beta_n^2(d, \varphi)$. We denote this average power gain by $G(d, \varphi)$.  

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The second is a large-scale variation due to changes in \( \{\beta_n(d, \varphi)\} \) and \( \{\tau_n(d, \varphi)\} \) — both in the number of paths and their strengths. These changes happen on the scale of the distance between objects in the environment.

To study these two scales of variation separately, we redraw Figure 3 in terms of two components as shown in Figure 4.

\[
s(t) \xrightarrow{x} \sqrt{G(d, \varphi)} \xrightarrow{c_{d,\varphi}(\xi)} y(t)
\]

Figure 4: Small-scale and large-scale variation components of channel

Here \( c_{d,\varphi} \) is normalized so that the average power gain introduced by \( c_{d,\varphi} \) is 1, i.e.

\[
c_{d,\varphi}(\xi) = \sum_n \beta_n(d, \varphi)e^{j\phi_n(d, \varphi)}\delta(\xi - \tau_n(d, \varphi))
\]

where \( \{\beta_n(d, \varphi)\} \) is normalized so that \( \sum_n \beta_n^2(d, \varphi) = 1 \). The large-scale variations in (average) amplitude gain are then lumped into the multiplicative term \( \sqrt{G(d, \varphi)} \).

The goal of wireless channel modeling is to find useful analytical models for the variations in the channel. Models for the large scale variations are useful in cellular capacity-coverage optimization and analysis, and in radio resource management (handoff, admission control, and power control). Models for the small scale variations are more useful in the design of digital modulation and demodulation schemes (that are robust to these variations). We hence focus on the small scale variations in this class.

**Small-scale Variations in Gain**

\[
s(t) \xrightarrow{x} \sqrt{G} \xrightarrow{c_{d,\varphi}(\xi)} y(t)
\]

Figure 5: Small-scale variations in the channel (with large-scale variations treated as constant).

Recall that the small scale variations in the channel are captured in a linear system with response

\[
c_{d,\varphi}(\xi) = \sum_n \beta_n(d, \varphi)e^{j\phi_n(d, \varphi)}\delta(\xi - \tau_n(d, \varphi))
\]
where the \( \{ \beta_n(d, \varphi) \} \) are normalized so that \( \sum_n \beta_n^2(d, \varphi) = 1 \). As \((d, \varphi)\) changes with \( t \), the channel corresponding to the small-scale variations becomes time-varying and we get:

\[
c(t; \xi) := c_{d(t), \varphi(t)}(\xi) = \sum_n \beta_n(t) e^{j\phi_n(t)} \delta(\xi - \tau_n(t)) .
\]

Treating the large scale variations \( \sqrt{G(d, \varphi)} \) as roughly constant (see Figure 5), we obtain:

\[
y(t) = \sqrt{G} \int_0^\infty c(t; \xi) s(t - \xi) d\xi .
\]

Finally, we may absorb the scaling factor \( \sqrt{G} \) into the signal \( s(t) \), with the understanding that the power of \( s(t) \) is the received signal power after passage through the channel. Then

\[
y(t) = \int_0^\infty c(t; \xi) s(t - \xi) d\xi .
\]

**Doppler shifts in phase**

For movements of the order of a few wavelengths, \( \{ \beta_n(t) \} \) and \( \{ \tau_n(t) \} \) are roughly constant, and the time variations in \( c(t; \xi) \) are mainly due to changes in \( \{ \phi_n(t) \} \), i.e.,

\[
c(t; \xi) \approx \sum_n \beta_n e^{j\phi_n(t)} \delta(\xi - \tau_n) .
\]

From this equation it is clear that the magnitude of the impulse response \( |c(t; \xi)| \) is roughly independent of \( t \). A typical plot of \( |c(t; \xi)| \) is shown in Figure 6. The width of the delay profile (delay spread) is of the order of tens of microseconds for outdoor channels, and of the order of hundreds of nanoseconds for indoor channels. Note that the paths typically appear in clusters in the delay profile (why?).

To study the phase variations \( \phi_n(t) \) in more detail, consider a mobile that is traveling with velocity \( v \) and suppose that the \( n \)-th path has an angle of arrival \( \theta_n(t) \) with respect to the velocity vector as shown in Figure 7. (Note that we may assume that \( \theta_n \) is roughly constant over the time horizon corresponding to a few wavelengths.) Then for small \( \Delta_t \),

\[
\phi_n(t + \Delta_t) - \phi_n(t) \approx \frac{2\pi f_c v \Delta_t \cos \theta_n}{c} = \frac{2\pi v \Delta_t}{\lambda_c} \cos \theta_n ,
\]

where \( \lambda_c \) is the carrier wavelength and \( c \) is the velocity of light. The frequency shift introduced by the movement of the mobile is hence given by

\[
\lim_{\Delta t \to 0} \frac{\phi_n(t + \Delta_t) - \phi_n(t)}{2\pi \Delta t} = \frac{v}{\lambda_c} \cos \theta_n = f_{\text{max}} \cos \theta_n ,
\]

where \( f_{\text{max}} = v/\lambda_c \) is called the maximum Doppler frequency. We will use this model for the variations in \( \phi_n(t) \) to characterize small-scale variations statistically in the following section.

**Definition 1.** The quantity \( \tau_{ds} = \max \tau_n(d, \varphi) - \min \tau_n(d, \varphi) \) is called the delay spread of the channel.

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Figure 6: Typical delay profile of channel with LOS path having delay 0.

Without loss of generality, we may assume that the delay corresponding to the first path arriving at the receiver is 0. Then \( \min \tau_n(d, \varphi) = 0, \tau_{ds} = \max \tau_n(d, \varphi) \), and:

\[
y(t) = \int_0^{\tau_{ds}} c(t; \xi)s(t - \xi)d\xi.
\]

Figure 7: Doppler Shifts
Frequency Nonselective (Flat) Fading

If the bandwidth of transmitted signal $s(t)$ is much smaller than $1/\tau_{ds}$, then $s(t)$ does not change much over time intervals of the order of $\tau_{ds}$. Thus

$$y(t) \approx s(t) \int_0^{\tau_{ds}} c(t, \xi) = s(t) \sum_n \beta_n e^{j\phi_n(t)}.$$

This implies that the multipath channel simply scales the transmitted signal without introducing significant frequency distortion. The variations with time of this scale factor are referred to as frequency nonselective, or flat, fading.

Note that the distortions introduced by the channel depend on the relationship between the delay spread of the channel and the bandwidth of the signal. The same channel may be frequency selective or flat, depending on the bandwidth of the input signal. With a delay spread of 10 $\mu$s corresponding to a typical outdoor urban environment, an AMPS signal (30 kHz) undergoes flat fading, whereas an IS-95 CDMA signal (1.25 MHz) undergoes frequency selective fading.

For flat fading, the channel model simplifies to

$$y(t) = V(t)s(t)$$

where

$$V(t) = \int_0^{\tau_{ds}} c(t, \xi)d\xi = \sum_n \beta_n e^{j\phi_n(t)}.$$

Purely Diffuse Scattering - Rayleigh Fading

Our goal is to model $\{V(t)\}$ statistically, but before we do that we distinguish between the cases where the multipath does or does not have a line-of-sight (LOS) component. In the latter case, the multipath is produced only from reflections from objects in the environment. This form of scattering is purely diffuse and can be assumed to form a continuum of paths, with no one path dominating the others in strength. When there is a LOS component, it usually dominates all the diffuse components in signal strength.

To model $\{V(t)\}$ statistically, we first fit a stochastic model to the phases $\{\phi_n(t)\}_{n=1,2,...}$.

**Assumption 1.** The phases $\{\phi_n(t)\}_{n=1,2,...}$ are well modeled as independent stochastic processes, with $\phi_n(t)$ being uniformly distributed on $[0, 2\pi]$ for each $t$ and $n$.

Using this assumption, we get the following results:

1. $\{V(t)\}$ is a zero-mean process. This is because

$$E[V(t)] = \sum_n \beta_n E[e^{j\phi_n(t)}] = 0$$

2. The process $\{v_n(t)\}$ defined by $v_n(t) = \beta_n e^{j\phi_n(t)}$ is a proper complex random process.
Proof. We need to show that the pseudocovariance function of \( \{v_n(t)\} \) equals zero.

\[
\hat{C}(t + \tau, t) = E[v_n(t)v_n(t + \tau)]
\]

\[
= E[(\beta_n e^{j\delta_n(t)}) (\beta_n e^{j\delta_n(t+\tau)})]
\]

\[
= \beta_n^2 E[e^{j(\delta_n(t) + \delta_n(t+\tau))}]
\]

\[
\approx \beta_n^2 E[e^{j(2\delta_n(t) + 2\pi f_{\text{max}} \tau \cos \theta_n)}] = 0
\]

where the approximation on the last line follows from the Doppler equation.

\[\square\]

③ If the number of paths is large, we may apply the Central Limit Theorem to conclude that \( \{V(t)\} \) is a proper complex Gaussian (PCG) random process.

First order statistics of \( \{V(t)\} \) for purely diffuse scattering

For fixed \( t \), \( V(t) = V_I(t) + jV_Q(t) \) is PCG random variable with

\[
E\left[|V(t)|^2\right] = \sum_n \beta_n^2 = 1 .
\]

Since \( V(t) \) is proper, \( V_I(t) \) and \( V_Q(t) \) are uncorrelated and have the same variance, which equals half the variance of \( V(t) \). Since \( V(t) \) is also Gaussian, \( V_I(t) \) and \( V_Q(t) \) are independent as well. Thus \( V_I(t) \) and \( V_Q(t) \) are independent \( \mathcal{N}(0, 1/2) \) random variables.

Envelope and Phase Processes

The envelope process \( \{\alpha(t)\} \) and the phase process \( \{\phi(t)\} \) are defined by

\[
\alpha(t) = |V(t)| = \sqrt{V_I^2(t) + V_Q^2(t)} , \quad \text{and} \quad \phi(t) = \tan^{-1}\left(\frac{V_Q(t)}{V_I(t)}\right) .
\]

We can write \( x(t) = V(t)s(t) \) in terms of \( \alpha(t) \) and \( \phi(t) \) as:

\[
x(t) = \alpha(t)e^{j\phi(t)}s(t) .
\]

This means that for flat fading, the channel is “seen” as a single path with gain \( \alpha(t) \) and phase shift \( \phi(t) \). Note that \( \alpha \) and \( \phi \) vary much more rapidly than the gain and phase of the individual paths \( \beta_n \) and \( \phi_n \) (why?).

For fixed \( t \), using the fact that \( V_I(t) \) and \( V_Q(t) \) are independent \( \mathcal{N}(0, 1/2) \) random variables, it is easy to show that \( \alpha(t) \) and \( \phi(t) \) are independent random variables with \( \alpha(t) \) having a Rayleigh pdf and \( \phi(t) \) being uniform on \([0, 2\pi]\). The pdf of \( \alpha(t) \) is given by

\[
p_\alpha(x) = 2xe^{-x^2/2}1_{\{x \geq 0\}} .
\]

It is easy to show that \( E[\alpha] = \sqrt{\pi/4} \) and \( E[\alpha^2] = E\left[|V(t)|^2\right] = 1 .\]

Since the envelope has a Rayleigh pdf, purely diffuse fading is referred to as Rayleigh fading.

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Scattering with a LOS component – Ricean Fading

If there is a LOS (specular) path with parameters $\theta_0$, $\beta_0$ and $\phi_0(t)$ in addition to the diffuse components, then
\[
V(t) = \beta_0 e^{j\phi_0(t)} + \sqrt{1 - \beta_0^2} \tilde{V}(t)
\]
where $\{\tilde{V}(t)\}$ is a zero mean PCG, Rayleigh fading process with variance $E[|\tilde{V}(t)|^2] = 1$.

**Note:** $\{\tilde{V}(t)\}$ is also a zero-mean process, but it is not Gaussian since the LOS component $\{\beta_0 e^{j\phi_0(t)}\}$ dominates the diffuse components in power. However, conditioned on $\{\phi_0(t)\}$, $\{V(t)\}$ is a PCG process with mean $\{\beta_0 e^{j\phi_0(t)}\}$.

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**Rice Factor:** The Rice factor $\kappa$ is defined by
\[
\kappa = \frac{\text{power in the LOS component}}{\text{total power in diffuse components}} = \frac{\beta_0^2}{1 - \beta_0^2}.
\]

From the definition of $\kappa$ it follows that
\[
\beta_0 = \sqrt{\frac{\kappa}{\kappa + 1}}, \quad \text{and} \quad 1 - \beta_0^2 = \frac{1}{(\kappa + 1)}.
\]

For fixed $t$, the pdf of the envelope $\alpha(t)$ can be found by first computing the joint pdf of $\alpha(t)$ and $\phi(t)$, conditioned on $\phi_0(t)$. This is straightforward since, conditioned on $\phi_0(t)$, $V(t)$ is a PCG random variable with mean $\beta_0 e^{j\phi_0(t)}$.

We can then show that the pdf of $\alpha(t)$ conditioned on $\phi_0(t)$ is not a function of $\phi_0(t)$, and we get:
\[
p_{\alpha|\phi_0}(x) = \frac{2x}{1 - \beta_0^2} I_0 \left( \frac{2x\beta_0}{1 - \beta_0^2} \right) \exp \left[ -\frac{x^2 + \beta_0^2}{1 - \beta_0^2} \right] 1_{\{x \geq 0\}} = p_{\alpha}(x)
\]

This pdf is called a Ricean pdf and it can be rewritten in terms of $\kappa$ as:
\[
p_{\alpha}(x) = 2x(\kappa + 1) I_0 \left( 2x\sqrt{\kappa(\kappa + 1)} \right) \exp \left[ -x^2(\kappa + 1) - \kappa \right] 1_{\{x \geq 0\}}.
\]

where $I_0(\cdot)$ is the zeroth order modified Bessel function of 1st kind, i.e.,
\[
I_0(y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp(y \cos \phi) d\phi.
\]
It is easy to see that when $\kappa = 0$, $p_\alpha(x)$ reduces to a Rayleigh pdf.

The pdf of $\alpha^2(t)$ is easily computed as:

$$p_\alpha^2(x) = \frac{p_\alpha(\sqrt{x})}{2\sqrt{x}} = (\kappa + 1) I_0 \left( 2\sqrt{x\kappa(\kappa + 1)} \right) \exp \left[ -x(\kappa + 1) - \kappa \right] \mathbb{1}_{\{x \geq 0\}}.$$ 

Figure 9: Ricean pdf for various Rice factors.

**Signaling Through Slow Flat Fading Channels**

We assume that the long-term variations in the channel are absorbed into $\mathcal{E}_m$. Then $\mathcal{E}_m$ represents the average received symbol energy (for symbol $m$) over the time frame for which the multipath profile may be assumed to be constant. Then the received signal is given by:

$$r(t) = V(t)s(t) + w(t) = \alpha(t)e^{j\phi(t)}s(t) + w(t)$$

where $\mathbb{E}[\alpha^2(t)] = 1$.

For slow fading, $\alpha(t)$ and $\phi(t)$ may be assumed to be constant over each symbol period. Thus, for memoryless modulation and symbol-by-symbol demodulation, $y(t)$ for demodulation over symbol period $[0, T_s]$ may be written as

$$r(t) = \alpha e^{j\phi} s_m(t) + w(t) \quad \text{(conditioned on symbol } m \text{ being transmitted)}$$

**Average probability of error for slow, flat fading**

The error probability is a function of the received signal-to-noise ratio (SNR), i.e., the received symbol energy divided by the noise power spectral density. We denote the symbol SNR by $\gamma_s$, and the corresponding bit SNR by $\gamma_b$, where $\gamma_b = \gamma/\nu$ and $\nu = \log_2 M$. 

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For slow, flat fading, the received SNR is 
\[ \gamma_s = \frac{\alpha^2 \mathcal{E}_s}{N_0}. \]
The average SNR (averaging over \( \alpha^2 \)) is given by 
\[ \overline{\gamma}_s = \mathbb{E}[\alpha^2] \frac{\mathcal{E}_s}{N_0} = \frac{\mathcal{E}_s}{N_0}. \]
The corresponding bit SNR’s are given by 
\[ \gamma_b = \frac{\gamma}{\nu}, \quad \text{and} \quad \overline{\gamma}_b = \frac{\mathcal{E}_b}{N_0 \nu} = \frac{\mathcal{E}_b}{N_0}. \]

Suppose the symbol error probability with SNR \( \gamma_s \) is denoted by \( P_e(\gamma_s) \). Then the average error probability (averaged over the fading) is
\[ P_e = \int_0^\infty P_e(x)p_{\gamma_s}(x)dx \]
where \( p_{\gamma_s}(x) \) is the pdf of \( \gamma_s \).

For Rayleigh fading, \( \alpha^2 \) is exponential with mean 1; hence \( \gamma_s \) is exponential with mean \( \overline{\gamma}_s \), i.e.,
\[ p_{\gamma_s}(x) = \frac{1}{\overline{\gamma}_s} \exp\left[-\frac{x}{\overline{\gamma}_s}\right] \mathbb{1}_{\{x \geq 0\}}. \]

For Ricean fading, \( \gamma_s \) has pdf
\[ p_{\gamma_s}(x) = \frac{\kappa + 1}{\overline{\gamma}_s} I_0 \left( 2\sqrt{\frac{\kappa(\kappa + 1)}{\overline{\gamma}_s}} \right) \exp\left[-\frac{x(\kappa + 1)}{\overline{\gamma}_s} - \kappa \right] \mathbb{1}_{\{x \geq 0\}}. \]

\( P_e \) for Rayleigh Fading

- Useful result (see problem 1 of HW#10):
\[ \int_0^\infty Q(\sqrt{x}) \frac{e^{-x/\gamma}}{\gamma} dx = \frac{1}{2} \left[ 1 - \sqrt{\frac{\gamma}{2+\gamma}} \right]. \]

BPSK
\[ P_b(\gamma_b) = Q(\sqrt{2\gamma_b}). \]
Thus we get
\[ P_b = \int_0^\infty Q(\sqrt{2x})p_{\gamma_b}(x)dx = \frac{1}{2} \left[ 1 - \sqrt{\frac{\gamma_b}{1+\gamma_b}} \right] \approx \frac{1}{4\gamma_b} \text{ (for large } \overline{\gamma}_b). \]
○ Binary coherent orthogonal modulation (e.g. FSK)

\[ P_b(\gamma_b) = Q(\sqrt{\gamma_b}) . \]

Here \( P_b \) is the same as that for BPSK with \( \tau_b \) replaced by \( \tau_b/2 \), i.e.,

\[ P_b = \frac{1}{2} \left[ 1 - \sqrt{\frac{\tau_b}{2 + \tau_b}} \right] \approx \frac{1}{2 \tau_b} \quad \text{(for large } \tau_b) . \]

○ Binary DPSK

\[ P_b(\gamma_b) = \frac{1}{2} e^{-\gamma_b} . \]

In this we case we may integrate directly to get

\[ P_b = \int_0^\infty \frac{1}{2} e^{-x} e^{-x/\tau_b} \frac{d\tau}{\gamma_b} = \frac{1}{2(1 + \tau_b)} \approx \frac{1}{2 \tau_b} \quad \text{(for large } \tau_b) . \]

○ Binary noncoherent orthogonal modulation (FSK)

\[ P_b(\gamma_b) = \frac{1}{2} e^{-\gamma_b/2} . \]

Here \( P_b \) is the same as that for DPSK with \( \tau_b \) replaced by \( \tau_b/2 \), i.e.,

\[ P_b = \frac{1}{2 + \tau_b} \approx \frac{1}{\tau_b} \quad \text{(for large } \tau_b) . \]

○ Similar expressions may be derived for other M-ary modulation schemes. Note that without fading the error probabilities decrease exponentially with SNR, whereas with fading the error probabilities decrease much more slowly with SNR (inverse linear in case of Rayleigh fading).

○ \( P_\epsilon \) for Ricean Fading

○ Direct approach: Compute \( P_\epsilon \) by direct integration of \( P_\epsilon(\gamma_b) \) w.r.t. \( \gamma_b \). This is cumbersome except in some special cases.

- Binary DPSK.

\[ P_b = \frac{\kappa + 1}{2(\kappa + 1 + \tau_b)} \exp \left[ -\frac{\kappa \tau_b}{\kappa + 1 + \tau_b} \right] . \]

as done in class. Again, it is easy to check that we get the Rayleigh result when \( \kappa = 0 \) and the AWGN result as \( \kappa \rightarrow \infty \).

- Binary noncoherent FSK. Same as DPSK with \( \tau_b \) replaced by \( \tau_b/2 \).

○ There are other approaches for computing expressions for the error probability for Ricean fading channels, which are beyond the scope of this class.
Diversity Techniques for Flat Fading Channels

- Performance with fading is considerably worse than without fading, especially when the fading is Rayleigh.
- Performance may be improved by sending the same information on many (independently) fading channels.
- For signaling on $L$ channels, the received signal on the $\ell$-th channel is:
  \[ r_\ell(t) = \alpha_\ell e^{j\phi_\ell} s_{m,\ell}(t) + w_\ell(t), \quad \ell = 1, 2, \ldots, L, \quad m = 1, 2, \ldots, M, \]
  where the noise $w_\ell(t)$ is assumed to be independent across channels.
- If $\{\alpha_\ell e^{j\phi_\ell}\}_{\ell=1}^L$ are independent, we get maximum diversity against fading.
- How do we guarantee independence of channels? By separating them either in time, frequency or space.
  - Frequency separation must be $\gg \frac{1}{\tau_{ds}}$, where $\tau_{ds}$ is the delay spread.
  - Time separation must be $\gg \frac{1}{f_{\text{max}}}$, where $f_{\text{max}}$ is the maximum Doppler frequency.
  - Spatial separation must be $> \frac{\lambda_c}{2}$, where $\lambda_c$ is the carrier wavelength.

Memoryless linear modulation with diversity

- When symbol $m$ is sent on the channels
  \[ r_\ell(t) = \alpha_\ell e^{j\phi_\ell} \sqrt{E_{s,\ell}} a_m e^{j\theta_m} g_\ell(t) + w_\ell(t), \quad \ell = 1, 2, \ldots, L, \quad m = 1, 2, \ldots, M, \]
  where $g_\ell(t)$ is a unit energy shaping function on channel $\ell$, $E_{s,\ell}$ is the average symbol energy on channel $\ell$, and the $a_m$’s are normalized so that $\frac{1}{M} \sum_m a_m^2 = 1$. We assume that the fading and noise are independent across channels. Note that $\{w_\ell(t)\}$ are independent PCG processes with PSD $N_0$.
- **Optimum receiver:** If we assume that the phases $\{\phi_\ell\}$ and the amplitudes $\{\alpha_\ell\}$ are estimated perfectly at the receiver, the optimum test statistic is formed by Maximal Ratio Combining (MRC) as
  \[ R = \sum_{\ell=1}^L \alpha_\ell \sqrt{E_{s,\ell}} e^{-j\phi_\ell} \langle r_\ell(t), g_\ell \rangle. \]
  We proved that his was optimum in class; also see Problem 4 of HW#10.
- The sufficient statistic $R$ may be rewritten as
  \[ R = \sum_{\ell=1}^L \alpha_\ell^2 \sqrt{E_{s,\ell}} a_m e^{j\theta_m} + \sum_{\ell=1}^L \alpha_\ell \sqrt{E_{s,\ell}} e^{-j\phi_\ell} W_\ell, \]
  where $\{W_\ell\}$ are independent $\mathcal{CN}(0, N_0)$ (note that the $e^{-j\phi_\ell}$ term multiplying $W_\ell$ does not matter since $W_\ell$ is circularly symmetric).
The MPE (ML) decision rule is the same as without diversity except that the constellation is scaled in amplitude based on the fading on the channels.

**Special Case: BPSK with diversity**

- The sufficient statistic in this case takes the form
  \[ R = \pm \sum_{\ell=1}^{L} \alpha_{\ell}^2 \mathcal{E}_{b,\ell} + W, \]
  where \( W = \sum_{\ell=1}^{L} \alpha_{\ell} \sqrt{\mathcal{E}_{b,\ell}} W_{\ell} \) is PCG with
  \[ \mathbb{E}[|W|^2] = N_0 \sum_{\ell=1}^{L} \mathcal{E}_{b,\ell} \alpha_{\ell}^2. \]

- The MPE decision rule for equal priors (or the ML decision rule) is to decide +1 (symbol 1) if \( R_I > 0 \), and -1 (symbol 2) if \( R_I < 0 \).

- For fixed \( \{\alpha_{\ell}\} \),
  \[ P_b = P\{\{R_I > 0\} \mid \{\text{symbol 2 sent}\}\} = P \left\{ W_I > \sum_{\ell=1}^{L} \alpha_{\ell}^2 \mathcal{E}_{b,\ell} \right\} = Q \left( \sqrt{\frac{2 \sum_{\ell=1}^{L} \alpha_{\ell}^2 \mathcal{E}_{b,\ell}}{N_0}} \right) = Q \left( \sqrt{2 \gamma_b} \right) \]
  where \( \gamma_{b,\ell} \) is the received bit SNR on the \( \ell \)-th channel, and \( \gamma_b = \sum_{\ell=1}^{L} \gamma_{b,\ell} \) is the total received bit SNR.

- The average BER is given by
  \[ \overline{P}_b = \int_{0}^{\infty} Q \left( \sqrt{2x} \right) p_{\gamma_b}(x) dx. \]
  Thus, we may evaluate \( \overline{P}_b \) by first finding the pdf \( p_{\gamma_b}(x) \).

- **Special case: Rayleigh fading.** If the fading is Rayleigh on all channels, then
  - Case 1: \( \gamma_{b,\ell} \)'s are distinct for \( \ell = 1, 2, \ldots, L \).
    \[ \overline{P}_b = \sum_{\ell=1}^{L} C_\ell \frac{1}{2} \left[ 1 - \frac{\sqrt{\gamma_{b,\ell}}}{1 + \sqrt{\gamma_{b,\ell}}} \right] \]
    where
    \[ C_\ell = \prod_{i \neq \ell} \frac{\gamma_{b,\ell}}{\gamma_{b,\ell} - \gamma_{b,i}}. \]
Case 2: \( \gamma_{b,\ell} \)'s are identical, i.e. \( \gamma_{b,\ell} = \frac{\gamma_b}{L} \) for all \( \ell \).

\[
\overline{P}_b = \left[ A \left( \frac{\gamma_b}{L} \right) \right]^L \sum_{\ell=0}^{L-1} \binom{L-1}{\ell} \left[ 1 - A \left( \frac{\gamma_b}{L} \right) \right]^{\ell}
\]

with

\[
A \left( \frac{\gamma_b}{L} \right) = \frac{1}{2} \left[ 1 - \sqrt{\frac{\gamma_b}{L + \gamma_b}} \right].
\]

For large \( \gamma_b \),

\[
A \left( \frac{\gamma_b}{L} \right) \approx \frac{L}{4\gamma_b} \quad \text{and} \quad 1 - A \left( \frac{\gamma_b}{L} \right) \approx 1.
\]

Thus

\[
\overline{P}_b \approx \left( \frac{L}{4\gamma_b} \right)^L \sum_{\ell=1}^{L} \binom{L-1}{\ell} \left( \frac{2L - 1}{L} \right). \]

Note that with diversity \( \overline{P}_b \) decreases at \( (\gamma_b)^{-L} \) which is a significant improvement over the inverse linear performance obtained without diversity. (See Figure 10.)

Figure 10: BPSK with diversity on Rayleigh fading channels.