Last time: 2 coupled W.G.s
fundamental mode: \( \hat{E}_{1.0} \)\( \sim \) \( E = U_{1.2} y \) \( e^{i\beta_{1.2} z} \).

Satisfy: \( \left( D^2 + k_{1.2}^2 \right) E_{1.2} = 0 \).

By assuming \( \hat{E}_1 = A_1(z) e^{i\beta_1 z} U_{1.1} y \) and using \( \hat{E}_2 = A_2(z) e^{i\beta_2 z} U_{2.1} y \)

perturbation theory based on a polarization density, we derived:
\[ \partial_z a_1(z) = \imath C_{12} e^{i(\beta_2 - \beta_1) z} a_2(z) \]
\[ \partial_z a_2(z) = \imath C_{21} e^{i(\beta_1 - \beta_2) z} a_1(z) \]

We can turn this into a matrix equation, but the matrix elements are \( z \)-dependent. To that end let:

\[
\begin{cases}
A_1(z) = a_1(z) e^{i\beta_2 z} \\
A_2(z) = a_2(z) e^{i\beta_1 z}
\end{cases}
\]

then the coupled mode equation for these two are:

Observe that: \( \partial_z A(z) = i\beta A(z) + \imath \beta^2 \partial_z A(z) \). Hence:

\[
\begin{align*}
\partial_z A_1(z) &= i\beta_1 A_1(z) + \imath C_{12} A_2(z) \\
\partial_z A_2(z) &= i\beta_2 A_2(z) + \imath C_{21} A_1(z)
\end{align*}
\]

\[ \Rightarrow \partial_z \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = iM \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \]
The coupling matrix is:

\[
\begin{bmatrix}
\beta_1 & C_{12} \\
C_{12} & \beta_2
\end{bmatrix}
\]

Solve this:

**Eigenvalues of** \(\bar{M}\): 
\[\text{det}(\bar{M} - \lambda I) = 0.\]

\[ (\beta_1 - \lambda)(\beta_2 - \lambda) - C_{12} C_{11} = 0. \]

\[ \lambda^2 - (\beta_1 + \beta_2) \lambda + \beta_1 \beta_2 - C_{12} C_{11} = 0. \]

\[ \lambda = \frac{\beta_1 + \beta_2}{2} \pm \sqrt{\left(\frac{\beta_1 + \beta_2}{2}\right)^2 - 4\beta_1 \beta_2 + 4C_{12} C_{11}} = \frac{\beta_1 + \beta_2}{2} \pm \sqrt{\Delta^2 + C_{12} C_{11}}. \]

\[ D = \frac{1}{2}(\beta_1 - \beta_2). \]

\[ \Delta = \sqrt{D^2 + 4C_{12} C_{11}} = \xi. \]

\[ \lambda_1 = \frac{\beta_1 + \beta_2}{2} \pm \xi \]

like two coupled pendulums.
Eigenvalues: \[ \begin{bmatrix} A_1^+ \\ A_2^+ \end{bmatrix} \] such that \[ (M - \lambda I) \begin{bmatrix} A_1^+ \\ A_2^+ \end{bmatrix} = 0 \]

It is easy to find that:

\[ \begin{pmatrix} \Delta + \tilde{\eta} & -C_{12} \\ C_{12} & \Delta - \tilde{\eta} \end{pmatrix} \begin{bmatrix} A_1^+ \\ A_2^+ \end{bmatrix} = 0. \]

\[ \begin{align*}
\frac{A_{1+}}{A_{2+}} &= -\frac{C_{12}}{\Delta - \tilde{\eta}} > 0, \\
\frac{A_{1-}}{A_{2-}} &= -\frac{C_{12}}{\Delta + \tilde{\eta}} < 0
\end{align*} \]

(\( \Delta < \tilde{\eta} \) always)

\[ \therefore \text{ } A_+ \text{ is in phase mode; } A_- \text{ is out of phase mode.} \]
Interference of $\tilde{A}_+$ with $\tilde{A}_-$ causes power to go back and forth between the WGs.

The "Center of mass" wavenumber gives the phase for the average motion of the $\tilde{A}_+$ and $\tilde{A}_-$ modes. The "$q$" relative wavenumber is the scale at which energy is being exchanged between the waveguides.