SOLUTIONS

Problem 1. Slab Waveguides

Consider the source free wave equations in inhomogeneous media.

\[
\begin{align*}
\nabla \times \mu_r^{-1} \nabla \times \mathbf{E} - k_0^2 \epsilon_r \mathbf{E} &= 0 \\
\nabla \times \epsilon_r^{-1} \nabla \times \mathbf{H} - k_0^2 \mu_r \mathbf{H} &= 0
\end{align*}
\]

(1)

In optics it is usually the case that \( \mu_r(r) = 1 \). When we further restrict \( \epsilon_r(r) = \epsilon_r(x) \), the problem becomes one dimensional. We can always choose to orient the coordinate system such that all \( y \) variations become zero, \( (\partial_y \to 0) \). Furthermore, due to phase matching, we know that \( (\partial_z \to ik_z) \).

Part a) Given the above, show from Equation (1) that \( E_y(x, z) \) is decoupled from \( E_x \) and \( E_z \). Write down the differential equation that governs \( E_y(x, z) \).

Solution: There are two ways to solve this and the next part, we can choose to use the matrix representation method of Homework #1, or use vector calculus identities. For the \( \mathbf{E} \) equation, we have:

\[
\nabla \times \nabla \times \mathbf{E} - k_0^2 \epsilon_r \mathbf{E} = 0
\]

Turn the \( \nabla \times \nabla \times \) operator into \( \nabla \times \nabla \times = \nabla \nabla \cdot - \nabla^2 \). The Laplacian operator is diagonal. We can simplify the gradient of divergence operator using the charge free condition, \( \nabla \cdot \epsilon_r(x) \mathbf{E} = 0 \).

\[
\begin{align*}
\nabla \cdot \epsilon_r(x) \mathbf{E} &= \epsilon_r(x) \nabla \cdot \mathbf{E} + (\nabla \epsilon_r(x)) \cdot \mathbf{E} \\
&= \epsilon_r(x) \nabla \cdot \mathbf{E} + E_x \partial_x \epsilon_r(x) \\
\nabla \cdot \mathbf{E} &= -\epsilon_r(x)^{-1} [\partial_x \epsilon_r(x)] E_x
\end{align*}
\]

(2)

Hence, the term \( \nabla \nabla \cdot \mathbf{E} \) reduces to:

\[
\nabla \nabla \cdot \mathbf{E} = \nabla \left\{ -\epsilon_r(x)^{-1} [\partial_x \epsilon_r(x)] E_x \right\}
\]

(3)

Since \( \partial_y \to 0 \), this term does not contain a \( y \)-component, neither is \( E_y \) appearing in the \( x \) or \( z \)-component of this term. Hence the \( E_y(x, z) \) is not coupled to the \( E_x \) and \( E_z \) components. The
The equation governing \( E_y \) is:

\[
( \nabla^2 + k_0^2 \epsilon_r ) E_y(x, z) = (\partial_x^2 + k_0^2 \epsilon_r - k_z^2) E_y(x, z) = 0 \tag{4}
\]

**Part b)** Similarly, show from Equation (1) that \( H_y(x, z) \) is decoupled from \( H_x \) and \( H_z \). Write down the differential equation that governs \( H_y(x, z) \). [10 %]

**Solution:** Using the matrix representation method.

\[
\nabla \times e_r^{-1} \nabla \times = \begin{bmatrix}
0 & -ik_z & 0 \\
-ik_z & 0 & -\partial_x \\
0 & \partial_x & 0
\end{bmatrix} \begin{bmatrix}
1 \\
e_r(x) \\
1/\epsilon_r(x)
\end{bmatrix} \begin{bmatrix}
0 & -ik_z & 0 \\
0 & -\partial_x & 0 \\
0 & \partial_x & 0
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\frac{k_z^2}{\epsilon_r(x)} & 0 & \frac{ik_z}{\epsilon_r(x)} \\
0 & \frac{k_z^2}{\epsilon_r(x)} & \frac{1}{\epsilon_r(x)} \partial_x \\
\frac{ik_z \partial_x}{\epsilon_r(x)} & 0 & -\partial_x \frac{1}{\epsilon_r(x)} \partial_x
\end{bmatrix}
\tag{5}
\]

Notice that in the above matrix, all the elements that link the y-component to the x and z-components are zero. Hence \( H_y(x, z) \) is decoupled from \( H_x \) and \( H_z \). We can read off the equation governing \( H_y \) as:

\[
\begin{bmatrix}
\epsilon_r(x) \partial_x & 0 & \partial_x \\
\partial_x & \epsilon_r(x) & 0
\end{bmatrix} \begin{bmatrix}
1 \\
\partial_x
\end{bmatrix} \begin{bmatrix}
k_z^2 \epsilon_r(x) \\
k_z^2 \epsilon_r(x)
\end{bmatrix} H_y(x, z) = 0 \tag{6}
\]

Alternatively, we use a brute force method that will involve many vector calculus identities. Consider the y-component of \( \nabla \times e_r^{-1} \nabla \times H \):

\[
\hat{y} \cdot \nabla \times e_r^{-1} \nabla \times H = \frac{1}{\epsilon_r(x)} \hat{y} \cdot \nabla \times \nabla \times H + \hat{y} \cdot \left( \nabla \frac{1}{\epsilon_r(x)} \right) \times \nabla \times H
\]

\[
= -\frac{1}{\epsilon_r(x)} \hat{y} \cdot \nabla^2 H + (\nabla \times H) \cdot \left[ \hat{y} \times \left( \nabla \frac{1}{\epsilon_r(x)} \right) \right]
\]

\[
= -\frac{1}{\epsilon_r(x)} \nabla^2 H_y - \left( \partial_x \frac{1}{\epsilon_r(x)} \right) \hat{z} \cdot \nabla \times H
\]

\[
= \frac{k_z^2}{\epsilon_r(x)} H_y - \frac{1}{\epsilon_r(x)} \partial_x^2 H_y - \left( \partial_x \frac{1}{\epsilon_r(x)} \right) \partial_x H_y
\]

\[
= \frac{k_z^2}{\epsilon_r(x)} H_y - \frac{1}{\epsilon_r(x)} \partial_x^2 H_y - \left( \partial_x \frac{1}{\epsilon_r(x)} \right) \partial_x H_y
\]

\[
\hat{y} \cdot \nabla \times e_r^{-1} \nabla \times H = \left[ \frac{k_z^2}{\epsilon_r(x)} - \partial_x \frac{1}{\epsilon_r(x)} \partial_x \right] H_y \tag{7}
\]

Combining with the terms \( k_0^2 H_y \), applying a minus sign and multiplying through by \( \epsilon_r \) gives us Equation (6). Please note the placement of the brackets tell us where to terminate the differentiations.
Comment: Both methods are acceptable, but it should be evident that the matrix representation method is much more expedient. However, this is only true for Cartesian coordinates, whereas the method using vector calculus identities are applicable in general. Also, when we want to apply boundary conditions at some bounding surfaces, the matrix representation breaks down. We must use the second method for these problems. Hence, it is beneficial to be comfortable with both.

The results you have proven from Parts a) and b) is essentially the decoupling of TE and TM to z waves for a general 1-D problem. The dielectric slab waveguide in the figure below is a special case.

Part c) Given $E_y(x,z)$ for the TE wave and $H_y(x,z)$ for the TM wave, write down the $H(x,z)$ and $E(x,z)$ for the TE and TM waves, respectively. [5 %]

Solution: We start from Maxwell’s equations:

$$i\omega\mu_0 H = \nabla \times (\hat{y}E_y) = \nabla E_y \times \hat{y} = \hat{z}\partial_x E_y - i k_z \hat{x} E_y$$

Hence the magnetic field of the TE wave is:

$$H_{TE}(x,z) = \hat{z} \frac{1}{i\omega\mu_0} \partial_x E_y - \hat{x} \frac{k_z}{\omega\mu_0} E_y$$  \hspace{1cm} (8)

Now, using duality principle, we can write down the electric field of the TM wave as:

$$E_{TM}(x,z) = \hat{z} i \frac{1}{\omega\epsilon_0\epsilon_r} \partial_z H_y + \hat{x} \frac{k_z}{\omega\epsilon_0\epsilon_r} H_y$$  \hspace{1cm} (9)

Figure 1: A dielectric slab waveguide. For all parts of this problem, assume the waveguide is symmetric.

Part d) Consider the $TE_n$ and $TM_n$ guided mode of the dielectric slab waveguide. As $\omega \rightarrow \infty$, what becomes of the E-field and H-field in the core of the waveguide? What about outside of the core?  \hspace{1cm} [5 %]
Solution: In the interior of the slab waveguide, the \( \partial_x \) is proportional \( k_x \) for the \( n \)-th guided mode. Recall from the graphical determination of \( k_x \) that these are bounded by the blow up of the tan or cot functions. Hence, while the frequency \( \omega \to \infty \), \( k_x \) stays constant, \( k_z \to \infty \). In Equation (8) and (9) from Part c), we see that:

\[
H^{\text{TE}}(x, z) \to -\hat{x} \quad \text{and} \quad E^{\text{TM}}(x, z) \to \hat{x}
\]  

With the electric and magnetic fields in the +\( \hat{y} \) direction for TM and TE cases, respectively, we see that the fields inside the waveguide core becomes quasi-TEM. In both cases, the wave is right handed, as it should be.

You saw in Homework #4 that as \( \omega \to \infty \), the field outside the waveguide core decays rapidly and approaches zero. Hence, combined with the argument above we see that outside of the waveguide core, the field tends to an exponentially decaying plane wave hugging the surface of the core.

Part e) What is a dielectric slab in the electrostatic limit? Consider the TM\(_0\) mode. As \( \omega \to 0 \), which direction does the E-field point? Does this make sense? Comment on why there is no cutoff for the TM\(_0\) mode. [5 %]

In the presence of a constant electric field, the dielectric slab becomes polarized, and energy is stored inside the slab. This is similar to the behavior of a capacitor. Intuitively, since the capacitor can trap electrostatic energy, a guided mode exists even at zero frequency. Consider the sketch in the figure below. With the E-field in the x-direction polarizing the dipoles in the dielectric slab, we see

![Figure 2: Polarized dipoles and fields for the TM\(_0\) mode. Left, one phase front. Right, variations in z-direction. (Blue for E-field, red for H-field.)](image)


that, on a phase front, the \( H \)-field is in the \( y \)-direction. When multiple phase fronts are considered, we see that both \( E_x \) and \( H_y \) varies in the \( z \)-direction. This depicts the TM\(_0\) mode. Hence, in the \( \omega \to 0 \) limit, E-field points in the \( x \)-direction and the TM\(_0\) mode becomes the electrostatic capacitor.

**Comment:** The above arguments are mostly qualitative, it can be shown easily from the guidance condition that as \( \omega \to 0 \), \( k x^2 \approx k_0^2 (\epsilon_1 - \epsilon) \) for the symmetric slab waveguide. Meanwhile, \( k_y^2 \approx k_0^2 \epsilon \).

Hence, for low contrast case, \( \epsilon_1 - \epsilon << \epsilon_1 \), the E-field tends toward the \( x \)-direction. When this is not the case, we see that \( E_x \) and \( E_z \) are always on the same order of magnitude.

What we discover here is similar to the weakly guided optical fiber case, where the lowest order mode is almost a TEM wave. When the contrast is high, we cannot expect the solution to approach a TEM wave.

**Part f)** Based on your results in the previous parts, argue why the TE\(_n\) mode is better confined than the TM\(_n\) mode. (**Hint:** Boundary condition of normal fields.)

**Solution:** As the hint suggests, we should consider the boundary condition for the normal field. From Part c) we see that the TM modes have \( E_x \) while the TE modes don’t. Normal electric displacement field should be continuous across a charge free surface. This holds true at the core-cladding interface. Hence we have:

\[
\epsilon E_x^{\text{cladding}} = \epsilon_1 E_x^{\text{core}} \tag{11}
\]

Since \( \epsilon_1 > \epsilon \), we see that \( E_x \) jumps to a higher value outside of the waveguide core. Since TM and TE modes have the same decay constant outside, this jump causes the field to be less confined in the core region. TE field does not suffer from this as \( H_x \) is continuous across the core-cladding interface due to constancy of \( \mu \).

**Comment:** The charge free condition, \( \nabla \cdot \epsilon E \) applies at the core-cladding interface despite there being polarization charges at the boundary.

**Problem 2 Optical fiber**

**Part a)** In the extremely low contrast optical fiber, the equation satisfied by the field is

\[
[\nabla^2 + k^2(x,y)] \psi(x,y) = 0 \tag{12}
\]

For the step index fiber, what kind of boundary conditions are satisfied by \( \psi \) at the core-cladding interface?

**Solution:** Given a differential equation, we can typically get the boundary condition by imposing the balance of derivatives in the equation. We separate the Laplacian operator into the divergence of the gradient:

\[
\nabla^2 \psi = \nabla \cdot \nabla \psi \tag{13}
\]
Since the other term in the equation \( k^2(x, y) \) must remain everywhere finite, we need to keep the above second derivative finite. Immediately that calls for two boundary conditions. Continuity must be satisfied by \( \psi \) at the interface where \( k^2 \) exhibit a finite jump discontinuity.

\[
\psi_1(x_-, y-) = \psi_2(x_+, y+) \tag{14}
\]

Here the \(-, + \) signs indicate above and below the interface. Also, from the procedure outlined in Homework 1, we see that by replacing the \( \nabla \) with \( \hat{n} \cdot \nabla \), we have:

\[
\hat{n} \cdot \nabla \psi_1 = \hat{n} \cdot \nabla \psi_2 \tag{15}
\]

Comment: As discussed in lecture, the above equation is satisfied by each component of the transverse E-field and H-field in the fiber (in fact, it is satisfied also by the longitudinal component). The two boundary conditions above can be linked physically with the continuity of tangential E and H-fields as well as the continuity of normal E and H-fields. The normal continuity condition comes from the low contrast approximation.

Part b) Write down the possible solutions in the core region and the cladding region for the above equation for a circular, low contrast, optical fiber. [10 %]

Solution: It is convenient to work with cylindrical coordinates. In this case Equation (12) becomes the Bessel’s equation. The possible solutions are the Bessel, Hankel or modified Bessel functions. We know that physically we want for a guided mode to be regular in the interior of the core, and to decay away from the interface in the cladding region. Assuming the radius of the core region to be \( a \), this leaves the choice of:

\[
\psi(\rho, \phi, z) = \begin{cases} 
A \ J_n(k_1 \rho) e^{i\phi + ikz}, & \rho < a \quad \text{Core} \\
B \ K_n(\alpha_2 \rho) e^{i\phi + ikz}, & \rho > a \quad \text{Cladding} 
\end{cases} \tag{16}
\]

Part c) Obtain the guidance condition for the modes of the low contrast optical fiber. [10 %]

Solution: We obtain the guidance condition by imposing the boundary conditions found in Part a). These are:

\[
A \ J_n(k_1 \rho) = B \ K_n(\alpha_2 \rho) \tag{17}
\]
\[
A \ k_1 J_n' (k_1 \rho) = B \ \alpha_2 K_n' (\alpha_2 \rho) \tag{18}
\]
The ratio of the two equations give:

\[
\frac{k_{1\rho} J_n'(k_{1\rho}a)}{J_n(k_{1\rho}a)} = \frac{\alpha_2 K_n'(\alpha_2 a)}{K_n(\alpha_2 a)}
\]  

(19)

Now we can use the recurrence relation for the Bessel and Modified Bessel functions:

\[
J_n'(x) = \mp J_{n\pm 1}(x) \pm \frac{n}{x} J_n(x)
\]  

(20)

\[
K_n'(x) = -K_{n\pm 1}(x) \pm \frac{n}{x} K_n(x)
\]  

(21)

These recurrence relation can be found from Abramowitz and Stegun. Plugging either the \(n + 1\) or \(n - 1\) choices we find two guidance conditions:

\[
\frac{k_{1\rho} J_n(k_{1\rho}a)}{J_n(k_{1\rho}a)} = \frac{\alpha_2 K_{n\pm 1}(\alpha_2 a)}{K_n(\alpha_2 a)}
\]  

(22)

\[
\frac{k_{1\rho} J_n(k_{1\rho}a)}{J_n(k_{1\rho}a)} = -\frac{\alpha_2 K_{n-1}(\alpha_2 a)}{K_n(\alpha_2 a)}
\]  

(23)

The two equations can be related by an addition or subtraction of \(2n/a\). Hence, the guided mode they determine are degenerate.

**Problem 3 Coupled mode theory**

**Part a)-d)** Do problem 8.12 of the textbook, Physics of Photonic Devices.

**Part a) solution:** The Fourier coefficients \(\Delta \epsilon_p(x)\) can be found by inverting the expression:

\[
\Delta \epsilon_p(x) = \epsilon_0 \sum_{p=-\infty}^{\infty} \Delta \epsilon_p(x) \exp \left(\frac{2\pi i pz}{\Lambda}\right)
\]  

(24)

This is an elementary exercise in Fourier series:

\[
\Delta \epsilon_p(x) = \frac{1}{\Lambda} \int_0^\Lambda dz \Delta(x, z) e^{-i\frac{2\pi pz}{\Lambda}}
\]  

(25)

The integral can be easily computed to give:

\[
\Delta \epsilon_p(x) = \begin{cases} 
0 & \text{if } p \text{ even} \\
\frac{2i}{\pi} p [\epsilon_1 - \epsilon] & \text{if } p \text{ odd}
\end{cases}
\]
Part b) solution: From Equation (8.5.11) in the textbook:

\[ K_{ab} = \frac{\omega \epsilon_0}{4} \int_{-\infty}^{+\infty} dx \Delta \epsilon_1(x)|E_y^{(0)}(x)|^2 \]  

(26)

Hence, we need to make use of the unperturbed solution for a slab waveguide. Due to the particular form of \( \Delta \epsilon_1(x) \), the above integral simplifies considerably:

\[ K_{ab} = \frac{i\omega}{2\pi \epsilon_0} (\epsilon_1 - \epsilon) \int_{-h}^{h} dx |E_y^{(0)}(x)|^2 \]  

(27)

We need to evaluate the integral for a normalized fundamental mode of the slab waveguide. For the dielectric slab, the normalization factor is tedious to calculate, it is OK to leave it as an unknown, \( C_0^2 \), since it is not required in the later problems. The result of the integral is, assuming a thickness \( d \) of the waveguide:

\[ K_{ab} = \frac{i\omega C_0^2}{2\pi \epsilon_0 (\epsilon_1 - \epsilon)} \left[ \frac{1 - e^{-2\alpha h}}{2\alpha} + \frac{1}{4k_x} \left( \sin(k_x d) + \sin(k_x(2h - d) + 2k_x h) \right) \right] \]  

(28)

Here \( k_x \) and \( \alpha \) are the transverse wavenumber and the decay constants in the interior and exterior of the waveguide, respectively. Since this is a contra-directional coupling scheme and the material is lossless, we have:

\[ K_{ab} = -K^*_{ba} \]  

(29)

In this case it is reduced to \( K_{ab} = K_{ba} \).

Part c) solution: Both of these derivations are given in the textbook Section 8.5.3, the results are:

\[ B(0) = \frac{-iK_{ab}\sin(ql)}{q\cos(ql) - i\Delta \beta \sin(ql)} A(0) \]  

(30)

\[ A(l) = \frac{iS}{\Delta \beta \sinh(Sl) + iS \cosh(Sl)} A(0) \]  

(31)

Here, \( q = iS \).

Part d) solution: The plot is identical to that given in Figure 8.29 of the textbook.

Part e) Discuss qualitatively how the square grating compare with a sinusoidal grating of the same period.

[c]: 10 %, all others 5 %

Solution: As can be seen in the above parts, the main characteristic of a grating is the coupling
coefficient $K_{ab}$. Let’s compare the two. For a pure sinusoidal grating:

$$\sin(\Lambda z)(\epsilon_1 - \epsilon) = \frac{i}{2} \left(e^{-i\Lambda z} - e^{i\Lambda z}\right)(\epsilon_1 - \epsilon)$$

Hence, the ±1 coefficients are proportional to $(\epsilon_1 - \epsilon)/2$. As compared to the square grating of the same period, we have:

$$\frac{1}{2} < \frac{2}{\pi}$$

Hence, the square grating couples stronger than the sinusoidal grating. This is because the square grating has a sharp edge which leads to enhanced scattering and therefore better coupling between the forward and backward traveling waves. The square grating will have larger reflection and wider stop-band when compared to its sinusoidal counterpart.