Reading Assignments:
Physics of Photonic Devices, Chapter 5

1. Transmission Line Analogy

Consider a lossy transmission line characterized by conductance G, capacitance C, inductance L and resistance R (All these quantities are per unit length). If we label the direction along the transmission line with z then by considering a small segment of the line we can write down, using KVL and KCL, the following equations:

$$\frac{dV}{dz} + (R - i\omega L)I = 0$$

$$\frac{dI}{dz} + (G - i\omega C)V = 0$$

Part a) Using the above coupled equations, derive the second order differential equations for I and V. Show that the general solutions are of the form $a_+e^{-\gamma z} + a_-e^{\gamma z}$. Find the term $\gamma$ and explain the physical meaning of the solutions.

Part b) Assume that there is only signal traveling in the +z direction on the transmission line, find the ratio between the voltage and current signals. This is the characteristic impedance of the transmission line.\(^1\)

Part c) We join two semi-infinite transmission lines of different characteristic impedance at $z = 0$. For $z > 0$ we have $Z_{01}$ and for $z < 0$, $Z_{02}$. Assume a wave propagating along the positive z direction incident upon the joining point. The total voltage will take the form:

$$V(z) = \begin{cases} 
    e^{\gamma_1 z} + \Gamma_v e^{-\gamma_1 z} & z < 0 \\
    T_v e^{\gamma_2 z} & z > 0 
\end{cases}$$

(1)

Write down the corresponding form of the total current on both sides of the joint (Hint: make use of the characteristic impedances). Using the continuity of current and voltages at the joint, find $\Gamma_v$ and $T_v$ in terms of the characteristic impedances. Do these equations look familiar? Please comment.

Part d) Now assume the currents are given on both sides of the transmission line in the form:

$$I(z) = \begin{cases} 
    e^{\gamma_1 z} + \Gamma_i e^{-\gamma_1 z} & z < 0 \\
    T_i e^{\gamma_2 z} & z > 0 
\end{cases}$$

(2)

Find the coefficients $\Gamma_i$ and $T_i$. Comment on your findings.

2. Problem 5.8 from the text.

\(^1\)You will notice a minus sign coming from our convention of using $-i\omega t$. Don't worry about this term as everything is in terms of ratios of the characteristic impedance
3. Antireflection Coating

We would like to design a thin antireflection coating layer onto a substrate with a refractive index \( n_2 \) such that the reflection coefficient for a plane wave incident from the air region (\( n_0 \)) at normal incidence is zero.

Part a) Find the refractive index \( n_1 \) of the coating layer in terms of \( n_0 \) and \( n_2 \).

Part b) The thickness of the anti-reflection layer in terms of the wavelength.

Part c) Consider the specific case of \( n_2 = 3.6 \) and free space wavelength 1.55\( \mu \)m, calculate \( n_1 \) and the thickness \( d \).

4. Geometrical optics series

In this problem we will consider the slab problem from another viewpoint. Consider the dielectric slab problem represented by the figure below:

**Figure 1:** Reflection and transmission through a dielectric slab.

Part a) Label the incidence and transmission angles at the 0-1 interface with \( \theta_{01} \) and \( \theta_{10} \) respectively. Write down the coefficients \( r_{01}, t_{01}, r_{10}, t_{10} \) using the Fresnel refraction formula for TE polarization. What is the relation between \( r_{01}, r_{10} \)?

Part b) Consider the following sketch in which the reflection and transmission coefficients are decomposed into summations over multiple reflections. Write down the summation for the reflection coefficient \( r = r^{(0)} + r^{(0)} + ... \) and show that there is a geometrical series embedded in the summation that represents multiple reflections. Perform the summation and show that your result agree with the results of matrix optics. Can you write down the transmission coefficient?

\[
r = \frac{r_{01} + r_{12} e^{i2k_1d}}{1 + r_{01}r_{12} e^{i2k_1d}}
\]  

(3)
Problem 1. Transmission line analogy:

a). \[ \begin{cases} \frac{dU}{dx} + (R - j \omega L) I = 0 \quad \text{(i)} \\ \frac{dI}{dx} + (G - j \omega C) V = 0 \quad \text{(ii)} \end{cases} \]

These are called "Telegaph equations."

\[ \frac{d^2 V}{dx^2} + (R - j \omega L) \frac{dI}{dx} = \frac{d^2 V}{dx^2} + (R - j \omega L) (j \omega L - G) V = 0. \]

\[ \therefore \frac{d^2 V}{dx^2} - \gamma^2 V = 0 \quad \text{(3.1)} \]

\[ \gamma = \sqrt{(R - j \omega L)(G - j \omega L)} \]

\[ \text{Similarly:} \]

\[ \frac{d^2 I}{dx^2} - \gamma^2 I = 0 \quad \text{(3.2)} \]

The general solution of (3.1) & (3.2) are both of the form:

\[ A_1 e^{-\gamma x} + A_2 e^{\gamma x} \]

We can imagine attaching the time dependence \( e^{-j\omega t} \), then:

\[ e^{-\gamma x} e^{-j\omega t} \] in an (attenuated) wave traveling in \(-x\) direction.

\[ e^{\gamma x} e^{j\omega t} \] in \(+x\) direction.

Thus we may use the notation:

\[ a - e^{-\gamma x} + a_+ e^{\gamma x} \]

b). Assume \( V = e^{\gamma x} \) (set \( a_+ = 1 \) without loss of generality).

\[ \frac{dV}{dx} = -(R - j \omega L) I \]

\[ \therefore I = -\frac{V}{(R - j \omega L)} \]

\[ \frac{1}{I} = \frac{1}{\gamma} \sqrt{\frac{G - j \omega C}{R - j \omega L}} \]

\[ \therefore \frac{V}{I} = -\frac{1}{\gamma} \sqrt{\frac{G - j \omega C}{R - j \omega L}} \]

we got a minus sign because it was "\(-xt\""
c) \[ V(z) = \begin{cases} e^{\gamma_1 z} + \frac{1}{T} e^{-\gamma_2 z} & z < 0 \\ \frac{1}{T} e^{\gamma_2 z} & z \geq 0 \end{cases} \]

To get the current, first consider starting from the equation (1) \[ \eta(z) \]

\[ I = -\frac{1}{(R - j\omega L)} \left\{ \frac{dV}{dz} \right\} \]

\[ z = 0: \quad I = -\frac{1}{(R - j\omega L)} \left\{ \gamma_2 e^{\gamma_2 z} - \frac{1}{T} e^{-\gamma_1 z} \right\} \]

\[ = \left[ \frac{\gamma_1}{(R - j\omega L)} \right] e^{\gamma_1 z} - \left[ \frac{\gamma_2}{(R - j\omega L)} \right] \frac{1}{T} e^{-\gamma_2 z} \]

\[ = \left[ \frac{\gamma_1}{Z_0} \right] e^{\frac{\gamma_1 z}{Z_0}} - \left[ \frac{\gamma_2}{Z_0} \right] \frac{1}{T} e^{-\frac{\gamma_2 z}{Z_0}} \]

\[ \therefore I = e^{\frac{\gamma_1 z}{Z_0}} - \frac{1}{T} e^{-\frac{\gamma_2 z}{Z_0}} \]

Notice the minus sign.

For \( z > 0 \): \[ I = \frac{1}{Z_0} e^{\gamma_2 z} \]

The voltage is a scalar quantity while the charge flow is certainly a vector. Projected onto \( 1-0 \) (circuit theory), we get the current \( I \) which is input a pseudo-scaler.

If you remember the current has a direction, you can directly use the characteristic impedances.

\[ z < 0: \quad I = e^{\frac{\gamma_1 z}{Z_0}} - \frac{1}{T} e^{-\frac{\gamma_2 z}{Z_0}} \]

\[ z > 0: \quad I = \frac{T}{Z_0} e^{\frac{\gamma_2 z}{Z_0}} \]
Voltage continuity: \[ V_{0-} = V_{0+} \Rightarrow 1 + T_v = T_v \]

Current continuity: \[ I_{0-} = I_{0+} \Rightarrow \frac{1}{Z_{01}} \left\{ (1 - T_i) \right\} = \frac{1}{Z_{02}} T_i \]

\[
\frac{Z_{01}}{Z_{02}} T_v + T_v = 2 \Rightarrow T_v = \frac{2 Z_{02}}{Z_{01} + Z_{02}}
\]

\[
T_i = \frac{Z_{02} - Z_{01}}{Z_{01} + Z_{02}}
\]

These look like the form of transmission, reflection coefficients for TE wave at normal incidence.

c1. \[ I(z) = \begin{cases} e^{\gamma_1 z + T_i} e^{-\gamma_1 z} & z < 0 \\ T_i e^{\gamma_2 z} & z > 0 \end{cases} \]

Again: \[ \frac{dI}{dz} = - (\gamma_1 - i \omega C) V. \]

\[ \infty : \gamma_1 e^{\gamma_1 z} - \gamma_1 T_i e^{-\gamma_1 z} = - (\gamma_1 - i \omega C) V. \]

\[ V = \left[ - \frac{\gamma_1}{\gamma_1 - i \omega C} \right] e^{\gamma_1 z} - \left[ - \frac{\gamma_1}{\gamma_1 - i \omega C} \right] T_i e^{-\gamma_1 z} \]

\[ \therefore V = Z_{01} e^{\gamma_1 z} - Z_{02} e^{-\gamma_1 z} T_i \]

\[ Z > 0 : V = Z_{02} e^{\gamma_2 z} T_i. \]

Voltage continuity: \[ V_{0-} = V_{0+} \Rightarrow Z_{01} - Z_{01} T_i = Z_{02} T_i \]

Current continuity: \[ I_{0-} = I_{0+} \Rightarrow 1 + T_i = T_i \]

\[
\frac{Z_{01}}{Z_{02}} T_i + T_i = 2 \Rightarrow T_i = \frac{2 Z_{01}}{Z_{01} + Z_{02}}
\]

\[
T_i = \frac{Z_{02} - Z_{01}}{Z_{01} + Z_{02}}
\]

IM TE at normal incidence!
Problem # 2.

Textbook 5.8:

a) \( d = 10 \mu m, \quad n = 3.5, \quad \lambda = 1 \mu m \)

Use formula (5.8.16) from text.

\[
\begin{align*}
\gamma &= \frac{r_0 + r_1 e^{i 2k \lambda d}}{1 + r_0 r_1 e^{i 2k \lambda d}} \\
t &= \frac{t_0 t_1 e^{i k \lambda d}}{1 + t_0 t_1 e^{i 2k \lambda d}}
\end{align*}
\]

For this special case: \( r_{01} = r_{21} = -r_{12} \)

\[
\begin{align*}
\gamma &= \frac{r_0 - r_{01} e^{2i k \lambda d}}{1 + r_{01}(-r_{01}) e^{2i k \lambda d}} = r_{01} \frac{1 - e^{2i k \lambda d}}{1 + r_{01}^2 e^{2i k \lambda d}} \\
t &= \frac{(1+r_{01})(1-r_{01}) e^{i k \lambda d}}{1-r_{01}^2 e^{i 2k \lambda d}} = 1 - r_{01}^2 e^{i 2k \lambda d}.
\end{align*}
\]

Free space wavelength: \( \lambda = 1 \mu m \).

in GaAs wavelength: \( \lambda_c = \frac{1}{3.5} \mu m \).

\( k_{ix} = \frac{2 \pi}{\lambda_i} = 2 \times 3.5 \pi = 7 \pi \).

\( i k_{ix} d = 2i k_{ix} d \)

\( e^{i k_{ix} d} = e^{2i k_{ix} d} = 1 \)

\( r_{01} = 0 \) and \( t = 1 \), which is quite expected since \( d = n \lambda \).
We also see that the condition for this to be true is

\[ kn \cdot d = N \lambda_0 \quad ; \quad N \in \text{integer}. \]

\[ \frac{n_1 2\pi}{\lambda_0} d = N 2\pi \]

\[ n_1 d = N \lambda_0 \]

\[ \downarrow \]

"Optical thickness (length)"

b) \[ R^{TE} = |r^{TE}|^2 = 0. \]

\[ T^{TE} = 1 - R^{TE} = 1 \]
Problem #3. Anti-reflection coating.

\[ a) \quad n = \frac{n_0 + r_{12} e^{i 2 \kappa x d}}{1 + r_{01} r_{12} e^{i 2 \kappa x d}} \]

\[ d = ? \quad n_i = ? \]

\[ n_0 = 1 \]

\[ e^{i 2 \kappa x d} = \pm 1 \]

1. For \( e^{i 2 \kappa x d} = -1 \) or \( \kappa x d = (m + \frac{1}{2}) \pi \), \( m \in \mathbb{N} \).

\[ R_{01} = R_{12} \quad \text{will give} \quad n = 0. \]

**TE case:**

\[ R_{01} = \frac{n_0 - n_1}{n_0 + n_1} = \frac{n_1 - n_2}{n_1 + n_2} = \frac{n_1}{n_1 + n_2} \]

**TM case:**

\[ R_{01} = \frac{n_1 - n_0}{n_1 + n_0} = \frac{n_2 - n_1}{n_2 + n_1} \]

\[ n_1 = \sqrt{n_0 n_2} \quad n_\parallel = \sqrt{n_0 n_2} \]

2. For \( e^{i 2 \kappa x d} = 1 \) or \( \kappa x d = m \pi \), \( m \in \mathbb{N} \).

\[ R_{01} = -R_{12} \quad \text{will give} \quad n = 0. \]

**TE case:**

\[ R_{01} = \frac{n_0 - n_1}{n_0 + n_1} = \frac{n_2 - n_1}{n_1 + n_2} = -R_{12} \]

\[ N_0 n_1 - n_1 n_2^2 + n_0 n_2 = n_0 n_2 - n_2^2 + n_2 - n_0. \]

\[ n_1 (n_0 - n_2) = n_1 (n_2 - n_0) \]

\[ n_1 = \text{anything} \]

\[ n_0 = n_2 \] (like the above case in problem #2).
TM case:

\[ n_0 = \frac{n_1 - n_0}{n_0 + n_1} = \frac{n_1 - n_2}{n_1 + n_2} = -n_{12} \]

No difference! (This is always true at normal incidence.)

\[ \text{If } k_{1x}d = (m + \frac{1}{2})\pi, \quad m \in \mathbb{N}. \]

\[ n_1 = \sqrt{n_2n_0} \]

\[ \text{If } k_{1x}d = m\pi, \quad m \in \mathbb{N}. \]

and if \( n_2 = n_0 \), \( n_1 \) = anything

b). We have two possibilities, the second one was covered in Problem #2.

Consider \( k_{1x}d = (m + \frac{1}{2})\pi, \quad m \in \mathbb{N}. \) \( n_1 = \sqrt{n_0n_2} \)

\[ k_{1x} = \frac{2\pi}{\lambda_1} = \frac{n_1\cdot2\pi}{\lambda_0} \]

\[ \therefore \frac{n_1d}{\lambda_0} = \frac{1}{2} \left( m + \frac{1}{2} \right) \pi \]

\[ n_1d = \lambda_0 \left( \frac{m}{2} + \frac{1}{4} \right) = \beta \left( \frac{\lambda}{2} \right) + \frac{1}{4} \lambda_0, \quad \beta \in \mathbb{N}. \]

\[ \therefore \text{The optical length } n_1d \text{ should be greater when!} \]
c) $n_2 = 3.6$ and $\lambda_0 = 1.55$

$\sqrt{n_1} = \sqrt{3.6} \approx 1.897$

$n_1 d = m \left(\frac{\lambda_0}{2}\right) + \frac{\lambda_0}{4}, m \in \mathbb{N}$

$d = \frac{1}{n_1}\left[m \left(\frac{\lambda_0}{2}\right) + \frac{\lambda_0}{4}\right], m \in \mathbb{N}$

$\therefore d \approx 0.41 m + 0.205$
Problem #4 Geometric Optics Series.

1. \( a \) \[
\begin{align*}
\alpha & = \theta_0 \\
\beta & = \theta_1 \\
\gamma & = \theta_0 \\
\delta & = \theta_1 \\
\end{align*}
\]

Fresnel equations for TE case:
\[
\begin{align*}
R & = \frac{n_1 \cos \theta_1 - n_2 \cos \theta_2}{n_1 \cos \theta_1 + n_2 \cos \theta_2} \\
T & = \frac{2n_1 \cos \theta_1}{n_1 \cos \theta_1 + n_2 \cos \theta_2}
\end{align*}
\]

For \( \theta = \theta_0 \) interface.

- \( i \to \text{medium} \ 0 \), \( \theta_i = \theta_0 \)
- \( t \to \text{medium} \ 1 \), \( \theta_t = \theta_0 \)

\[
\begin{align*}
R_{01} & = \frac{n_0 \cos \theta_0 - n_1 \cos \theta_1}{n_0 \cos \theta_0 + n_1 \cos \theta_1} \\
T_{01} & = \frac{2n_0 \cos \theta_0}{n_0 \cos \theta_0 + n_1 \cos \theta_1}
\end{align*}
\]

For \( \theta = \theta_0 \) interface.

- \( i \to \text{medium} \ 1 \), \( \theta_i = \theta_0 \)
- \( t \to \text{medium} \ 0 \), \( \theta_t = \theta_0 \)

Obviously \(-R_{01} = T_{01}\)

\[
T_{10} = \frac{2n_1 \cos \theta_0}{n_1 \cos \theta_0 + n_0 \cos \theta_0}
\]

or \( T_{10} = 1 + R_{01} = 1 - R_{01} \).
\[ r^{(0)} = n_0, \text{ obviously.} \]

b) First order:

\[ \text{Transmission} \quad \frac{d}{k_{ix}} \rightarrow \text{reflection} \quad \frac{d}{k_{ix}} \rightarrow \text{transmission} \]

\[ r^{(1)} = t_{10} e^{ik_{ix} d} r_{1z} e^{ik_{ix} d} t_{10}. \]

\[ \therefore r^{(1)} = t_{10} r_{1z} t_{10} e^{2ik_{ix} d}. \]

Second order:

\[ r^{(2)} = t_{10} e^{ik_{ix} d} r_{1z} e^{ik_{ix} d} t_{10} e^{ik_{ix} d} r_{1z} e^{ik_{ix} d} t_{10}. \]

\[ \therefore r^{(2)} = t_{10} r_{1z} t_{10} e^{2ik_{ix} d} r_{10} r_{1z} e^{2ik_{ix} d}. \]

We see that the \( n+1 \) order is obtained from the \( n \)-th order by the process of \( V_0 = V_{10} r_{1z} e^{2ik_{ix} d} \)

\[ \therefore r^{(n)} = t_{10} r_{1z} t_{10} e^{2ik_{ix} d} \left[ r_{10} r_{1z} e^{2ik_{ix} d} \right]^{n-1} \]

\[ \therefore r = r^{(0)} + r^{(1)} + r^{(2)} + \ldots \]

\[ r = r_{01} + \sum_{n=1}^{\infty} t_{10} r_{1z} t_{10} e^{2ik_{ix} d} \left[ r_{10} r_{1z} e^{2ik_{ix} d} \right]^{n-1} \]

\[ = r_{01} + t_{10} r_{1z} t_{10} e^{2ik_{ix} d} \frac{1 + \left[ r_{10} r_{1z} e^{2ik_{ix} d} \right]}{1 - r_{10} r_{1z} e^{2ik_{ix} d}} \left\rceil \text{This must go to zero,} \right. \]

\[ \therefore r = r_{01} + \frac{r_{1z} e^{-ik_{ix} d} t_{10} t_{10}}{1 - r_{10} r_{1z} e^{2ik_{ix} d}} \]
using the relations: \( t_{01} = 1 + r_{01} \) \( \& \) \( r_{10} = -r_{01} \), we have:

\[
t_{10} = 1 - r_{21}
\]

\[
\gamma = \gamma_{01} + \frac{(1+r_{01})(1-r_{21})e^{ikrd}}{1+r_{01}r_{21}e^{2ikrd}} r_{12} = \gamma_{01} + \frac{r_{12} e^{ikrd}}{1+r_{01}r_{12} e^{2ikrd}}
\]

\[
\therefore \gamma = \gamma_{01} + \frac{r_{12} e^{ikrd}}{1+r_{01}r_{12} e^{2ikrd}}
\]

Now for the transmission coefficient:

\[
t^{(0)} = t_{01} e^{ikrd}
\]

\[
t^{(i)} = t_{01} e^{ikrd} r_{12} e^{ikrd} r_{10} e^{ikrd} t_{21}
\]

so now we just odd terms like "Af" which is also

\[
r_{12} r_{10} e^{2ikrd}
\]

\[
\therefore t = t^{(0)} + t^{(1)} + \ldots
\]

\[
= \sum_{n=0}^{\infty} t_{01} e^{ikrd} t_{21} [ r_{12} r_{10} e^{2ikrd} ]^n
\]

\[
t = \frac{t_{01} t_{21} e^{ikrd}}{1+r_{01}r_{12} e^{2ikrd}}
\]

\[
t_{12} = 1 + r_{12}
\]

\[
t_{01} = 1 + r_{01}
\]

\[
r_{10} = -r_{01}
\]

\[
 \therefore t = \frac{(1+r_{12})(1+r_{01}) e^{ikrd}}{1+r_{01}r_{12} e^{2ikrd}} e^{ikrd}
\]