Reading Assignments:
Physics of Photonic Devices, Sections 2.1, 5.1 - 5.4

1. Some mathematics of fields

Part a) In physics, we can roughly understand that a field is a machinery that takes as input (argument) a position in space and time and produces an output. In electromagnetics we deal with scalar fields and vector fields, whose outputs are scalars and vectors, respectively. Let's first review some very basic properties of these fields.
- A vector multiplying a scalar produces a _______ vector______
- A row vector multiplying a column vector produces a _______ scalar_____. This operation is sometimes called a(an) _______ inner________ product.
- A _______ matrix________ maps a column vector to another column vector.
- A column vector multiplying a row vector produces a _______ matrix_______. This operation is sometimes called a(an) _______ outer_______ product.

Part b) Differential operators.
- The gradient operator acts on a _______ scalar________ field and produces a _______ vector_______ field.
- The divergence operator acts on a _______ vector________ field and produces a _______ scalar_______ field.
- The curl operator acts on a _______ vector________ field and produces a _______ vector_______ field.

Part c) Representation of operators, a useful trick.
If a scalar field is represented as a scalar function of space $F(x, y, z)$, then a vector field can be conveniently represented as a column vector in Cartesian space. $F = \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix}$. Where $F_x, F_y, F_z$ are three scalar functions of space. A column vector can represent the gradient operator $\nabla = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix}$.

Take a moment and convince yourself this is true by referring back to the previous part. Then, write down the representation of the divergence and curl operators. What is the Laplacian operator $\nabla^2$ on a scalar field. What is the Laplacian on vector fields.

Part d) Vector identities
Use this formalism developed above and the rules of matrix multiplication to show that for any (sufficiently differentiable) fields $A$ and $F$ the following holds:
i) $\nabla \cdot (\nabla \times A) = 0$
ii) $\nabla \times \nabla F = 0$
iii) $\nabla \times (\nabla \times A) = \nabla (\nabla \cdot A) - \nabla^2 A$
iv) $\nabla \times (FA) = (\nabla F) \times A + F \nabla \times A$
Problem #1 cont.

Part C.

From the answers to Part B we can argue that:

\[ \nabla f(x,y,z) = \overrightarrow{F}(x,y,z). \]

\[
\text{scalar} \quad \text{vector}
\]

hence this must be a vector.

Since \( \nabla f = \left( \frac{\partial}{\partial x} \hat{x} + \frac{\partial}{\partial y} \hat{y} + \frac{\partial}{\partial z} \hat{z} \right) f(x,y,z) = \left( \frac{\partial}{\partial x} \right) f(x,y,z), \)

short-hand \( \partial_x \)

A convenient representation for the gradient operator is:

\[ \nabla = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right). \]

Similarly:

Divergence: \( \nabla \cdot \overrightarrow{G}(x,y,z) = g_f(x,y,z), \) hence "\( \nabla \)" must be a row vector.

\[ \text{vector} \quad \text{scalar} \]

(Column) \quad \text{The transpose operation.}

\[ \nabla^T = (\partial_x, \partial_y, \partial_z) = \nabla^T \]

so divergence is gradient transpose.

Curl: \( \nabla \times \overrightarrow{H}(x,y,z) = \overrightarrow{J}(x,y,z). \) like in magnetostatics.

\[ \text{vector} \quad \text{vector} \]

must be a matrix.
To construct this matrix, consider the usual way of evaluating the curl:

\[
\nabla \times \mathbf{H} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} = \hat{x} \left( \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \right) + \hat{y} \left( \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \right) + \hat{z} \left( \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right).
\]

Arrange into a column vector fashion:

\[
\begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} \begin{pmatrix} H_x \\ H_y \\ H_z \end{pmatrix} = \begin{pmatrix} \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \\ \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \\ \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \end{pmatrix}
\]

Matrix:

We can read off:

\[
\begin{bmatrix} \nabla \times \end{bmatrix} = \begin{bmatrix} 0 & -\frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix}; \quad \text{curl} = \nabla \times \mathbf{H},
\]

which is the matrix representation of the curl.

The Laplacian operator on scalar fields is

\[
\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}
\]

short for \( \frac{\partial^2}{\partial x^2} \), etc.

Notice that this is a scalar operator, which would be the product of a row and vector operator and a column vector operator.

\[
\nabla^2 = \nabla \cdot \nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}
\]

For vector fields, the Laplacian acts on each scalar component.
Part d)

i) \( \nabla \cdot \nabla \cdot \vec{A} = 0 \).

Turn to matrix notation

\[
\begin{pmatrix}
d_x & d_y & d_z \\
d_y & 0 & -d_x \\
d_z & d_x & 0
\end{pmatrix}
\begin{pmatrix}
A_x \\ A_y \\ A_z
\end{pmatrix}
\]

Matrix multiplication is associative, so we can evaluate first.

\[
\begin{pmatrix}
d_x & d_y & d_z \\
d_y & 0 & -d_x \\
d_z & d_x & 0
\end{pmatrix}
\begin{pmatrix}
0 & -d_z & 2d_x \\
-2d_y & 0 & 2d_z \\
d_y & d_z & 0
\end{pmatrix}
\begin{pmatrix}
A_x \\ A_y \\ A_z
\end{pmatrix}
\]

With the assumption of differentiability on the field \( \vec{A} \), all these terms vanish.

\[
\therefore \nabla \cdot \nabla \cdot \vec{A} = 0.
\]

ii) \( \nabla \times \nabla F = 0 \).

\[
\begin{pmatrix}
0 & -d_z & 2d_x \\
d_z & 0 & -d_x \\
d_y & d_z & 0
\end{pmatrix}
\begin{pmatrix}
\partial x \\ \partial y \\ \partial z
\end{pmatrix}
\]

\[
\begin{pmatrix}
2d_x^2 - d_x d_y - d_z d_y \\
d_x d_z - d_d d_x - d_y d_z \\
2d_y^2 - d_y d_x - d_z d_y
\end{pmatrix}
\]

= 0 \text{ if } F \text{ satisfies the differentiability condition}

\[
\therefore \nabla \times \nabla F = 0.
\]
\[ \nabla \times \nabla \times \vec{A} = \nabla \nabla \cdot \vec{A} - \nabla^2 \vec{A} \]

\[ \begin{bmatrix} 0 - \frac{\partial}{\partial y} \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} 0 - \frac{\partial}{\partial x} \\ -\frac{\partial}{\partial z} \frac{\partial}{\partial y} 0 \end{bmatrix} \begin{bmatrix} 0 - \frac{\partial}{\partial z} \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} 0 - \frac{\partial}{\partial x} \\ -\frac{\partial}{\partial z} \frac{\partial}{\partial y} 0 \end{bmatrix} = \begin{bmatrix} -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} \\ \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2} \\ \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2} \end{bmatrix} \]

An outer product formed

\[ \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} = \nabla \nabla \cdot \vec{A} - \nabla^2 \vec{A} \]

\[ \nabla \times \nabla \times \vec{A} = \nabla \nabla \cdot \vec{A} - \nabla^2 \vec{A} \]

\[ \nabla \times (\nabla \cdot \vec{A}) = (\nabla \times \vec{A}) \cdot \nabla + \nabla \cdot (\nabla \times \vec{A}) \]

\[ \begin{bmatrix} 0 - \frac{\partial}{\partial z} \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} 0 - \frac{\partial}{\partial x} \\ -\frac{\partial}{\partial z} \frac{\partial}{\partial y} 0 \end{bmatrix} F \begin{bmatrix} \frac{A_x}{A_y} \\ \frac{A_y}{A_z} \\ \frac{A_z}{A_x} \end{bmatrix} , \text{ chain rule: } \nabla \cdot F = \nabla \frac{A_x}{A_y} + \frac{\partial}{\partial x} \frac{A_x}{A_y} \]

\[ = \begin{bmatrix} 0 - \frac{\partial F}{\partial y} \frac{\partial}{\partial z} + \frac{\partial^2}{\partial x^2} F \\ \frac{\partial F}{\partial z} 0 - \frac{\partial}{\partial x} \frac{\partial}{\partial y} F \\ -\frac{\partial F}{\partial y} \frac{\partial}{\partial z} 0 \end{bmatrix} \begin{bmatrix} \frac{A_x}{A_y} \\ \frac{A_y}{A_z} \\ \frac{A_z}{A_x} \end{bmatrix} \]

\[ = F \nabla \times \vec{A} + (\nabla \cdot \vec{A}) \vec{A} \]
\[ \nabla \times \mathbf{E} = -\partial_t \mathbf{B} \quad (1) \]
\[ \nabla \times \mathbf{H} = \partial_t \mathbf{D} + \mathbf{J} \quad (2) \]
\[ \nabla \cdot \mathbf{B} = 0 \quad (3) \]
\[ \nabla \cdot \mathbf{D} = \rho \quad (4) \]

2. Maxwell's equations

**Part a)** Give the names of equations (1) to (4) and give the units of all quantities appearing in them. If the sources are assumed known, how many unknown functions are there in Maxwell's equations? How many scalar differential equations are there in Maxwell's equations? (Again, it may be helpful to think of Problem 1 Part c)

**Part b)** Give the name and physical meaning of the equation below.

\[ \nabla \cdot \mathbf{J} + \partial_t \rho = 0 \quad (5) \]

Using equation (2) and (5) it is possible to derive a weaker form of equation (4) under one condition. What is this condition? Furthermore, what can we say to restrict the weaker form of (4) to equation (4)?

**Part c)** Suppose there are magnetic charge density \( \lambda \) and current density \( \mathbf{M} \) and that magnetic charges are conserved\(^1\), how should we augment equation (1) and (3)?

**Part d)** From Part b) we see that the four Maxwell's equations are not all independent in electrodynamics. How many independent scalar differential equations are there in Maxwell's equations? What must be given in order that the equations be solvable?

**Part e)** Search online and provide an example of a dispersive material. What about an anisotropic material. What are some applications of anisotropic materials in optics.

3. Boundary conditions

**Part a)** Problem 2.1 of Physics of Photonic Devices. [10 points]

**Part b)** Based on your answer to Part a), consider how we can obtain the boundary conditions of Maxwell's equations following the simple steps listed below\(^2\). Consider an interface between two materials:

i) In equations (1) and (2) set all surface quantities (unit \( \alpha \ m^{-2} \)) to zero.

ii) Replace \( \nabla \) with surface normal \( \mathbf{n} \) and take the difference of the left hand sides across the interface.

iii) In equations (3) and (4) set all volume quantities (unit \( \alpha \ m^{-3} \)) to zero.

iv) Replace \( \nabla \) with \( \mathbf{n} \) and take the difference of the left hand sides across the interface.

4. Wave equation

**Part a)** Augment equation (1) with magnetic current \( \mathbf{M} \) and together with equation (2) derive the wave equation for an inhomogeneous isotropic medium.

**Part b)** Use the duality principle (consult section 5.1.2 of Physics of Photonic Devices) and write down the corresponding wave equation for magnetic field. Check that you get the same result by deriving it 'directly'.

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\(^1\)You may think this is nonsense but in fact in some engineering applications it is convenient to introduce magnetic current densities.

\(^2\)In doing this problem please remember the possibility of surface current densities and surface charge densities, and their units!
Problem 12

Part a). Consult any textbook for names and units.

There are, apparently, 12 unknown scalar functions (fields) in Maxwell's equations. Three for $E$, $H$, $D$, $B$, respectively.

There are 8 scalar first order differential equations,

3 each for \( \nabla \times E = -\frac{\partial B}{\partial t} \)

1 each for \( \nabla \cdot E = 0 \)

\( \nabla \cdot B = \rho \).

Part b). From $\nabla \times E = \nabla \times B + J$ and $\nabla \times B + \partial t \rho = 0$,

Consider $\nabla \cdot (\nabla \times B + \nabla \times J) = 0$

\[ \nabla \cdot \left( \nabla \times B - \partial t \rho \right) = 0 \]

Switch the order of these two

\[ \nabla \cdot B - \partial t \rho = \text{const} \]

We may argue that at the beginning of time, $\nabla \cdot B - \partial t \rho = 0$ is obeyed, then it is obeyed throughout time.

Note that the above derivation doesn't work when $\partial t = 0$, or the static case. In statics, electrodynamics $\nabla \cdot B = \rho$ and magnetostatics $\nabla \times H = J$, are decoupled.
Part c). Convolved means: $\nabla \cdot \mathbf{M} = -\nabla \cdot \mathbf{E}$, like for electric charges.

If we define, $\mathbf{D} = \nabla \cdot \mathbf{E}$, in parallel with $\nabla \cdot \mathbf{B} = \mathbf{P}$, then,

$$2\pi \nabla \cdot \mathbf{E} - 2\pi \lambda = 0 = \nabla \cdot \mathbf{D} + \nabla \cdot \mathbf{M} = 0.$$

$$\therefore \nabla \cdot (2\pi \mathbf{B} + \mathbf{M}) = 0.$$

$$2\pi \mathbf{B} + \mathbf{M}$$ is some divergenceless function, or

$$2\pi \mathbf{B} + \mathbf{M} = \nabla \times \mathbf{F}$$

is an unknown field.

When $\mathbf{M} = 0$, $\mathbf{D} = \mathbf{E}$, so $\mathbf{F} = -\mathbf{D}$.

The consistent way to augment Faraday's law is:

$$\nabla \times \mathbf{E} = -\nabla \times \mathbf{B} - \mathbf{M}.$$

Part d). With the removal of $\nabla \cdot \mathbf{B} = \mathbf{P}$ and $\nabla \times \mathbf{E} = 0$ in electrodynamics, there are 6 independent equations left.

12 unknown with 6 equation, the problem is not solvable, we must have 6 more equations, which are the constitutive relations.

Part e). All materials are in fact dispersive,

Most materials are anisotropic, some are especially strongly anisotropic, such as graphene.

We can use anisotropic material for greater wear plates, polarizers, 3D film glasses.
Problem 2.3.

Part a) Follow textbook 2.1.2.

Part b) The essential idea is that we use the integral form of Maxwell's equations to obtain the boundary conditions, and as the integration volume/surface approach the boundary, the volume/surface quantities, or quantities with units of m\(^{-3}\) or m\(^{-2}\), respectively, vanish. Also, differential operators become differences across the boundary; hence, we get the Boundary Conditions by:

Start with Maxwell's equations:

\[ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \]
\[ \nabla \times \vec{H} = \vec{J} \]

Set all surface quantities to 0: \( \vec{E}_s = 0, \vec{H}_s = 0 \) and \( \vec{J} \to \vec{J}_s \) for surface current.

Set all volume quantities to 0: \( \vec{E}_v = 0, \vec{H}_v = 0 \) and \( \vec{J} \to \vec{J}_v \) for surface charge density.

Replace \( \nabla \times \) with \( \hat{n} \times \) and take the difference across the interface:

\[ \hat{n} \times \vec{E}_s = \hat{n} \times \vec{E}_v \]
\[ \hat{n} \times \vec{H}_s = \hat{n} \times \vec{H}_v + \vec{J}_s \]

and

\[ \hat{n} \cdot \vec{B}_s = \hat{n} \cdot \vec{B}_v \]
\[ \hat{n} \cdot \vec{D}_s = \hat{n} \cdot \vec{D}_v + \rho_s \]

* Note that the two normal boundary conditions, since they derive from the two divergence equations, are indeed not independent from the two tangential boundary conditions.
Part c) This problem is more challenging than the rest. Suppose that $\mu$ and $\epsilon$ are only functions of $z$. Derive the wave equation for $E_z$. Using techniques in Problem 1 will make this derivation simple. You may assume the source free condition in this problem.

Part d) Did you make use of the boundary conditions when deriving the above equation? Assume that $E_z$ varies in time as $\exp(-i\omega t)$, you should known why this assumption is valid, discuss how this wave equation for $E_z$ reflects the boundary conditions of the $E$ field.
Problem #4:

Part a). The consistent augmentation of (1) is:

\[
\begin{align*}
\nabla \times \bar{E} &= -\partial_t \bar{B} - \bar{M}, \\
\nabla \times \bar{H} &= \partial_t \bar{E} + \bar{J}.
\end{align*}
\]

For inhomogeneous isotropic medium, \( \mu = \mu(\bar{E}) \) and \( \varepsilon = \varepsilon(\bar{E}) \).

We derive the wave-equation for \( \bar{E} \)-field:

\[
\frac{1}{\mu} \nabla \times \nabla \times \bar{E} = -\partial_t \nabla \times \bar{H} - \nabla \times \frac{1}{\mu} \bar{M}
\]

\[
= -\partial_t \nabla \times \left\{ \mu \frac{\nabla \times \bar{H}}{\mu} \right\} - \nabla \times \frac{1}{\mu} \bar{M}
\]

\[
= -\partial_t \frac{\varepsilon}{\varepsilon(\bar{E})} \bar{E} - \partial_t \bar{J} - \nabla \times \frac{1}{\mu} \bar{M}
\]

\[
= \nabla \times \frac{1}{\mu} \nabla \times \bar{E} + \partial_t \varepsilon(\bar{E}) \bar{E} = -\partial_t \bar{J} - \nabla \times \frac{1}{\mu} \bar{M}.
\]

Part b). The duality principle:

\[
\begin{align*}
\bar{E} &\rightarrow \bar{P} \\
\bar{B} &\rightarrow -\bar{E} \\
\bar{\varepsilon} &\rightarrow \mu \\
\bar{\mu} &\rightarrow -\bar{\varepsilon}
\end{align*}
\]

\[
\therefore \quad \nabla \times \frac{1}{\varepsilon} \nabla \times \bar{E} + \nabla \times \frac{1}{\mu} \bar{P} = -\partial_t \bar{\varepsilon} + \nabla \times \frac{1}{\varepsilon} \bar{J}.
\]

It is easy to derive this from the procedure of part a).
Part c).

We shall simplify $\nabla \times \nabla \times \mathbf{E}$ with the knowledge that $\mu = \mu(\mathbf{E})$.

Consider problem 4.1, part d), iv): $\nabla \times (E \mathbf{A}) = (\nabla \times \mathbf{A}) E + E \nabla \times \mathbf{A}$.

Let $E = \frac{1}{\mu}$ and $\mathbf{A} = \mathbf{\times E}$, we have:

$\nabla \times \mathbf{E} = (\frac{1}{\mu}) \mathbf{\times} (\nabla \mathbf{E}) + \frac{1}{\mu} \nabla \times \nabla \times \mathbf{E}$.

$\mathbf{\times} \mathbf{E} + \frac{1}{\mu} \nabla \times \nabla \times \mathbf{E}.$

The $\mathbf{\times} \mathbf{E}$ term reveals that this quantity is perpendicular to the $\mathbf{E}$ direction, hence, it does not affect the equation of $E_z$.

Think about why this is, refer to the boundary conditions for $E_z$:

$\nabla \times \mathbf{E}_1 - \nabla \times \mathbf{E}_2 = 0.$

$\mathbf{n} \cdot \mathbf{E}_1 - \mathbf{n} \cdot \mathbf{E}_2 = 0.$

Notice how the $\mathbf{E}$-field is insensitive to discontinuities in $\mu$.

Since $\mu$ only changes in the $\mathbf{E}$-direction, it does not affect $E_z$, which is the normal $\mathbf{E}$-field in this case.

For $E_z$, only the second term matters

$\frac{1}{\mu} \nabla \times \nabla \times \mathbf{E}.$

Since the medium is source free, $\nabla \cdot \mathbf{D} = 0$, or $\nabla \cdot \mathbf{E} = 0$.

$\nabla \cdot \mathbf{E} = (\nabla \cdot \mathbf{E}) \cdot \mathbf{E} + \varepsilon (\nabla \times \mathbf{E}),$ which you can easily derive.
The \((\mathbf{E})\cdot \mathbf{E}\) term only has contribution from \(E_z\),

\[
(\mathbf{E}_x, \mathbf{E}_y, \mathbf{E}_z) \cdot \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = (0, 0, 2z \cdot E_z) \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = (2z \cdot E_z) E_z.
\]

\[
0 = (2z \cdot E_z) E_z + \varepsilon \cdot (\nabla \cdot \mathbf{E})
\]

\[
\nabla \cdot \mathbf{E} = -\frac{1}{\varepsilon} \partial_z (2z \cdot E_z).
\]

\[
\nabla \times \mathbf{E} = -\begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \frac{1}{\varepsilon} \partial_z (2z \cdot E_z). \text{ whose } E_z \text{ component is:}
\]

\[
- \partial_z \frac{1}{\varepsilon} \partial_z (2z \cdot E_z).
\]

So finally we have:

\[
(- \partial_z \frac{1}{\varepsilon} (2z \cdot E_z) - \nabla^2 + \partial_z^2 \mu \varepsilon \cdot E_z = 0.
\]

\[
(- \partial_x^2 - \partial_y^2 + \partial_z^2 \mu \varepsilon) E_z - \left( \partial_z \frac{1}{\varepsilon} (2z \cdot E_z) + \partial_z^2 \right) E_z = 0.
\]

This can be further simplified:

\[
\partial_z \frac{1}{\varepsilon} (2z \cdot E_z) + \partial_z^2 = \partial_z \left[ \frac{1}{\varepsilon} \partial_z (2z \cdot E_z) + \partial_z^2 \right] = \partial_z \left[ \frac{1}{\varepsilon} 2z \cdot E_z \right]
\]

from the chain rule.

The wave equation for \(E_z\) then:

\[
\left( -\varepsilon \partial_z \frac{1}{\varepsilon} \partial_z - \partial_x^2 - \partial_y^2 + \varepsilon \mu \partial_z^2 \right) \varepsilon E_z = 0.
\]
Part d).
In our derivations, we haven't used the B.C.'s explicitly. Of course, but if we look at

\[-\varepsilon \varepsilon ( \frac{1}{2} ) r - 2x^2 - 2y^2 + \varepsilon r^2 x^2 \] \( \varepsilon E_z = 0 \)

we can immediately read off that

\( E_z \) should be continuous in the \( x, y \) direction, which in to say that \( \text{transverse} \ E_z \) is continuous other wise, the second derivatives will be infinite. Also, the transverse derivatives of \( E_z \) are continuous which is a statement about the \( \vec{E} \) field, in fact.

For the normal component:

\( \varepsilon \varepsilon E_z \) can't be infinite, that's to say, \( \varepsilon E_z \varepsilon = \partial_z \)

is continuous in the \( z \)-direction. But, sudden jumps
in \( \varepsilon \varepsilon E_z \varepsilon \) is allowed since they can be offset by \( \frac{1}{2} \)
from \( \varepsilon \varepsilon E_z \varepsilon \) making \( \frac{1}{2} \varepsilon \varepsilon E_z \varepsilon \) continuous, again.

So the B.C.s are in fact embedded inside the differential equations and can be read off directly.