Reading Assignments:
Physics of Photonic Devices, Sections 2.1, 5.1 - 5.4

1. Some mathematics of fields

Part a) In physics, we can roughly understand that a field is a machinery that takes as input (argument) a position in space and time and produces an output. In electromagnetics we deal with scalar fields and vector fields, whose outputs are scalars and vectors, respectively. Let’s first review some very basic properties of these fields.
- A vector multiplying a scalar produces a ____________.
- A row vector multiplying a column vector produces a ____________. This operation is sometimes called a(an) ____________ product.
- A ____________ maps a column vector to another column vector.
- A column vector multiplying a row vector produces a ____________. This operation is sometimes called a(an) ____________ product.

Part b) Differential operators.
- The gradient operator acts on a ____________ field and produces a ____________ field.
- The divergence operator acts on a ____________ field and produces a ____________ field.
- The curl operator acts on a ____________ field and produces a ____________ field.

Part c) Representation of operators, a useful trick.
If a scalar field is represented as a scalar function of space $F(x,y,z)$, then a vector field can be conveniently represented as a column vector in Cartesian space. \( \mathbf{F} = \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} \). Where $F_x, F_y, F_z$ are three scalar functions of space. A column vector can represent the gradient operator $\nabla = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix}$.

Take a moment and convince yourself this is true by referring back to the previous part. Then, write down the representation of the divergence and curl operators. What is the Laplacian operator $\nabla^2$ on a scalar field. What is the Laplacian on vector fields.

Part d) Vector identities
Use this formalism developed above and the rules of matrix multiplication to show that for any (sufficiently differentiable) fields $\mathbf{A}$ and $F$ the following holds:

i) $\nabla \cdot \nabla \times \mathbf{A} = 0$

ii) $\nabla \times \nabla F = 0$

iii) $\nabla \times \nabla \times \mathbf{A} = \nabla \nabla \cdot \mathbf{A} - \nabla^2 \mathbf{A}$

iv) $\nabla \times (F \mathbf{A}) = (\nabla F) \times \mathbf{A} + F \nabla \times \mathbf{A}$
\[ \nabla \times E = -\partial_t B \quad (1) \]
\[ \nabla \times H = \partial_t D + J \quad (2) \]
\[ \nabla \cdot B = 0 \quad (3) \]
\[ \nabla \cdot D = \rho \quad (4) \]

2. Maxwell’s equations

Part a) Give the names of equations (1) to (4) and give the units of all quantities appearing in them. If the sources are assumed known, how many unknown functions are there in Maxwell’s equations? How many scalar differential equations are there in Maxwell’s equations? (Again, it may be helpful to think of Problem 1 Part c))

Part b) Give the name and physical meaning of the equation below.

\[ \nabla \cdot J + \partial_t \rho = 0 \quad (5) \]

Using equation (2) and (5) it is possible to derive a weaker form of equation (4) under one condition. What is this condition? Furthermore, what can we say to restrict the weaker form of (4) to equation (4)?

Part c) Suppose there are magnetic charge density \( \lambda \) and current density \( M \) and that magnetic charges are conserved\(^1\), how should we augment equation (1) and (3)?

Part d) From Part b) we see that the four Maxwell’s equations are not all independent in electrodynamics. How many independent scalar differential equations are there in Maxwell’s equations? What must be given in order that the equations be solvable?

Part e) Search online and provide an example of a dispersive material. What about an anisotropic material. What are some applications of anisotropic materials in optics.

3. Boundary conditions

Part a) Problem 2.1 of Physics of Photonic Devices. [10 points]

Part b) Based on your answer to Part a), consider how we can obtain the boundary conditions of Maxwell’s equations following the simple steps listed below\(^2\). Consider an interface between two materials:

i) In equations (1) and (2) set all surface quantities (unit \( \propto m^{-2} \)) to zero.

ii) Replace \( \nabla \) with surface normal \( \hat{n} \) and take the difference of the left hand sides across the interface.

iii) In equations (3) and (4) set all volume quantities (unit \( \propto m^{-3} \)) to zero.

iv) Replace \( \nabla \) with \( \hat{n} \) and take the difference of the left hand sides across the interface.

4. Wave equation

Part a) Augment equation (1) with magnetic current \( M \) and together with equation (2) derive the wave equation for an inhomogeneous isotropic medium.

Part b) Use the duality principle (consult section 5.1.2 of Physics of Photonic Devices) and write down the corresponding wave equation for magnetic field. Check that you get the same result by deriving it ‘directly’.

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\(^1\) You may think this is nonsense but in fact in some engineering applications it is convenient to introduce magnetic current densities.

\(^2\) In doing this problem please remember the possibility of surface current densities and surface charge densities, and their units!
Part c) This problem is more challenging than the rest. Suppose that $\mu$ and $\epsilon$ are only functions of $z$. Derive the wave equation for $E_z$. Using techniques in Problem 1 will make this derivation simple. You may assume the source free condition in this problem.

Part d) Did you make use of the boundary conditions when deriving the above equation? Assume that $E_z$ varies in time as $e^{\exp(-i\omega t)}$, you should known why this assumption is valid, discuss how this wave equation for $E_z$ reflects the boundary conditions of the $E$ field.