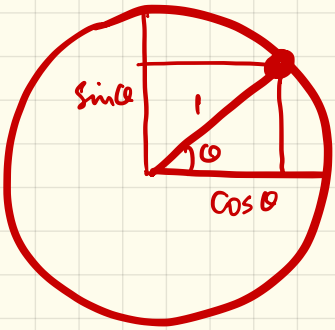
DSP:
Fourier Transform

- Model multiplicity $\longrightarrow$ count / numbers.
- Model opposite $\longrightarrow$ I have \$5
  How much should I add to make \$0? Ans: -5

Numbers & neg. numbers are all abstractions of reality.

Now, How to model rotation?



We have $\cos\theta$ and $\sin\theta$.
Of course we need to express their relation in some way. We cannot just say

$\cos\theta + \sin\theta$ ✗

or $\langle \cos\theta, \sin\theta \rangle$

Which still does not allow algebra.

OK, we can say $\vec{x}\cos\theta + \vec{y}\sin\theta$

But we don't have to say $\vec{x}$, we just need to make $\vec{y}$ become $\perp$ to $\vec{x}$.

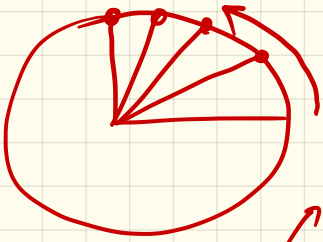
How? Well, one way is to say that $\vec{y}\cdot\vec{y}$ should be $-1.\vec{x}$.

i.e., $y^2 = -1$ $\quad\therefore y = \sqrt{-1} = j$

$\therefore$ Rotation is a vector of $\cos\theta + j\sin\theta$

Aha, this is exactly $= e^{j\theta}$ $\Rightarrow$ models rotation

(→) Now, what would rotation at different freq. look like?



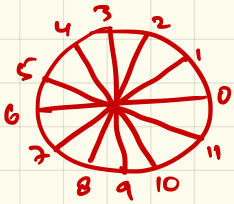
Say we want to model every freq. in N discrete steps.

One cycle/s

$$
\begin{array}{c}
 \\ n=0 \\ n=1 \\ n=2 \\ \vdots \\ n=N-1
\end{array}
\begin{array}{c}
\text{freq } 0 \\
\begin{bmatrix} e^{j0} \\ e^{j0} \\ e^{j0} \\ \vdots \\ e^{j0} \end{bmatrix}
\end{array}
\begin{array}{c}
\text{freq. } f_1 \\
\begin{bmatrix} e^{j0\cdot\frac{2\pi}{N}} \\ e^{j1\cdot 2\pi/N} \\ e^{j2\cdot 2\pi/N} \\ \vdots \\ e^{j(N-1)\cdot\frac{2\pi}{N}} \end{bmatrix}
\end{array}
\begin{array}{c}
\text{freq } 2f \\
\begin{bmatrix} e^{j0\cdot\left(2\left(\frac{2\pi}{N}\right)\right)} \\ e^{j1\cdot\left(2\cdot\frac{2\pi}{N}\right)} \\ e^{j2\left(2\cdot\frac{2\pi}{N}\right)} \\ \vdots \\ e^{j(N-1)\left(2\cdot\frac{2\pi}{N}\right)} \end{bmatrix}
\end{array}
\cdots
\begin{array}{c}
\text{freq } mf \\
\begin{bmatrix} e^{j0\left(m\frac{2\pi}{N}\right)} \\ e^{j\cdot 1\cdot\left(m\frac{2\pi}{N}\right)} \\ e^{j\cdot 2\cdot\left(m\frac{2\pi}{N}\right)} \\ \vdots \\ e^{j(N-1)\left(m\frac{2\pi}{N}\right)} \end{bmatrix}
\end{array}
\begin{array}{c}
\text{freq } (N-1)f \\
\vdots
\end{array}
$$

Fourier observed each column is orthogonal.

$f_0 \cdot f_1 = 0 \Rightarrow$ you can immediately see because

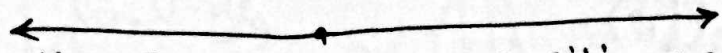


0	cancels	6
1	cancels	7
2	"	8
3	"	9
4	"	10
5	"	11

Same holds for any pair.

FOURIER MATRICES ⟶ SIGNAL PROCESSING

- Negative nos. don't exist in reality ; it's an abstraction.



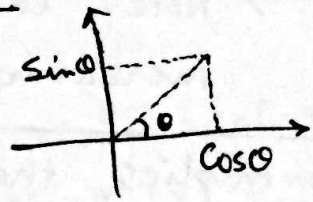
Negative $\mathbb{R}$ is imagination that helps math — 0 — positive real line is reality

- Complex # $(a+jb)$ also such an abstraction.
  - Physical interpretation of j is $90°$ rotation.

  - So $(a+jb)$ is a on Real axis and b after $90°$ rotation.
  - This is why (j^2) is (-1), $(j^3) = -i$ which is $90°$ rotation in the opposite direction.

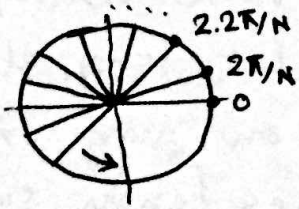
~~Now consider a stick rotating.~~

- Now, what is $e^{j\theta}$?
  - It's $= \cos\theta + j\sin\theta$. ⟶



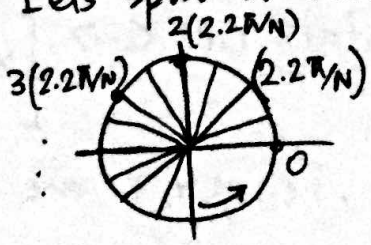
  - As θ changes, we get a circle.

- Enumerate the values of $e^{j\theta}$ (spinning stick) at different points during spin.
  - Say ~~spinning~~ we enumerate at granularity of $\frac{2\pi}{N}$ slices.



$$\Rightarrow \begin{bmatrix} e^{j0} \\ e^{j2\pi/N} \\ e^{j2.2\pi/N} \\ \vdots \\ e^{j(N-1)2\pi/N} \end{bmatrix}$$

  - Let's spin a little faster, say double.



$$\Rightarrow \begin{bmatrix} e^{j0} \\ e^{j2.2\pi/N} \\ e^{j2(2.2\pi/N)} \\ \vdots \\ e^{j(N-1)(2.2\pi/N)} \end{bmatrix} \Rightarrow \begin{bmatrix} e^{0} \\ e^{2(m2\pi/N)} \\ e^{3(m2\pi/N)} \\ \vdots \\ e^{(N-1)(m2\pi/N)} \end{bmatrix}$$

⊖ In general: $e^{j\left(m \cdot \frac{2\pi}{N}\right) n}$

$$
\begin{array}{c}
n=0 \leftarrow \\
n=1 \leftarrow \\
n=2 \leftarrow \\
\vdots \\
n=N-1 \leftarrow
\end{array}
\begin{bmatrix}
e^{0} & e^{0} & \cdots & e^{0} \\
e^{\left(\frac{2\pi}{N}\right)} & e^{\left(2\left(\frac{2\pi}{N}\right)\right)} & & e^{\left((N-1)\frac{2\pi}{N}\right)} \\
e^{2\cdot\left(\frac{2\pi}{N}\right)} & e^{2\left(2\frac{2\pi}{N}\right)} & & e^{2\left((N-1)\frac{2\pi}{N}\right)} \\
\vdots & \vdots & & \vdots \\
e^{(N-1)\left(\frac{2\pi}{N}\right)} & e^{(N-1)\left(2\cdot\frac{2\pi}{N}\right)} & & e^{(N-1)\left((N-1)\frac{2\pi}{N}\right)}
\end{bmatrix}
$$

$$
\begin{array}{cccc}
\downarrow & \downarrow & \cdots & \downarrow \\
m=1 & m=2 & \cdots & m=(N-1)
\end{array}
$$

⊖ This matrix is $N \times N$.
AND, any column of this matrix $\perp$ any other column.

↳ Note: each column corresponds to an increasing freq. of movement / spinning.

⊖ $\boxed{\text{This implies, that the different frequencies form a basis of the space of frequencies.}}$

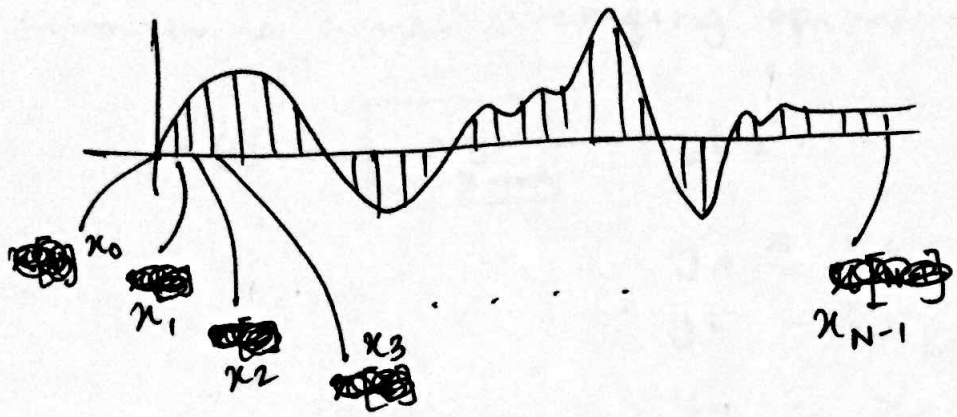
↳ i.e., each column is a vector denoting freq "m".

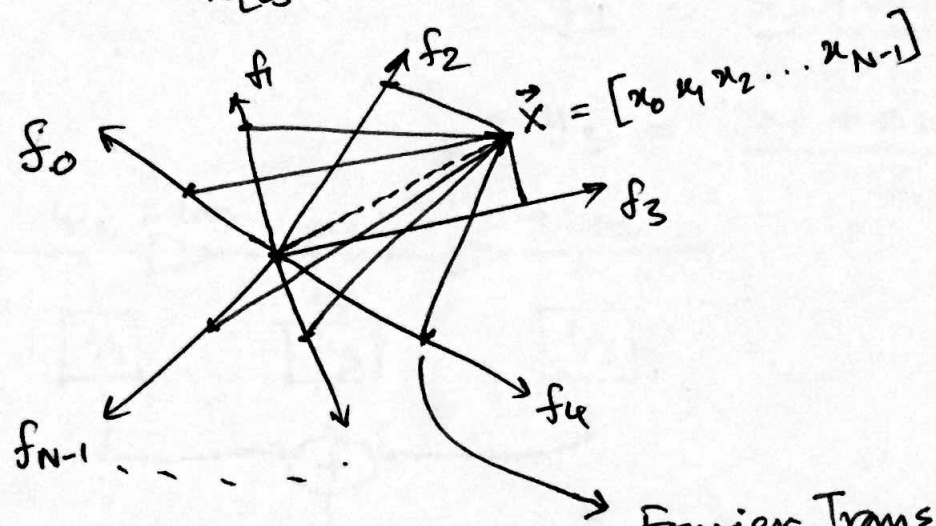
⊖ Now, when you have data, say GPS location
<lat, long, elevation>
You can project on any orthogonal basis, and the projected values (when summed) completely capture all info. in that data.

↳ i.e., ~~projected values~~ $GPS\langle\cdots\rangle = GPS\langle\cdots\rangle \cdot [x] + GPS[y] + GPS[z]$

So long as $x \perp y \perp z$, i.e., x, y, z are bases.

Now any data can be treated as vectors.





Fourier Transform of $X(t)$ or $X[n]$.

By Parseval's analysis Identity:

i.e.,

$$\boxed{X(m) = \sum_{n=0}^{N-1} x_n \cdot e^{-j\frac{2\pi}{N} m.n}}$$

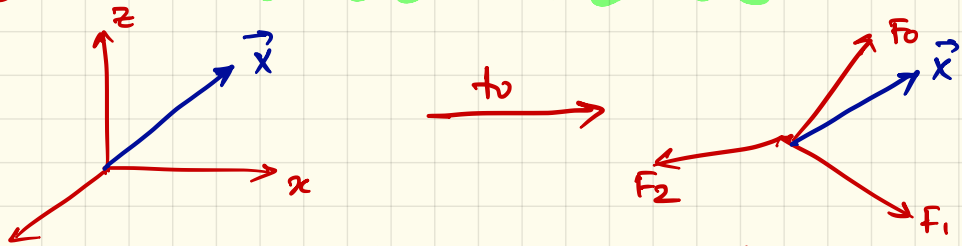
Projection of the $\vec{X}$ vector to the mth column of the matrix (or m^{th} frequency).

Google

- IDFT
- Cos + j Sin signal
- Cos 2πft signal
- Mag + phase
- Why symmetric/cyclic/conj.
- Time shift = phase shift
- Real fs, fs/N, fund. freq.
- Impact of fs/N

→ Diag.
⇒ PCA

Thus DFT is a change of basis:



This basis is the I matrix

$$\begin{array}{c} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ \begin{array}{ccc} x & y & z \end{array} \end{array}$$

This is the Fourier basis matrix F

$$\begin{array}{c} \begin{bmatrix} 1 & \cdot & \cdot \\ 1 & \cdot & \cdot \\ 1 & \cdot & \cdot \end{bmatrix} \\ \begin{array}{ccc} f_0 & f_1 & 2f_1 \end{array} \end{array}$$

Scaling by $\frac{1}{\sqrt{N}}$

Note that dot product = Projection only when the dot product is normalized.

Now, for DFT, observe that each column of F is not a unit vector, e.g., $[1\ 1\ 1]^T$.

$\therefore$ The DFT needs to be scaled by the length of the vector $1/\sqrt{N}$

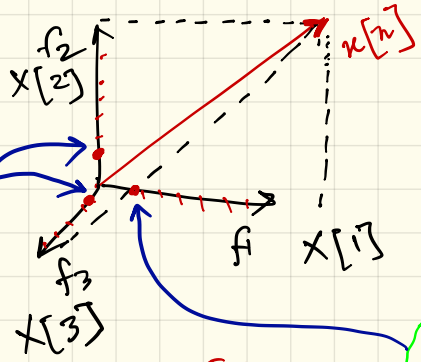
i.e.,

$$\text{DFT} \quad X_m = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}mn}$$

$$\text{IDFT} \quad x_n = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} X_m e^{j\frac{2\pi}{N}mn}$$

IDFT: Basically apply Parseval's synthesis identity:

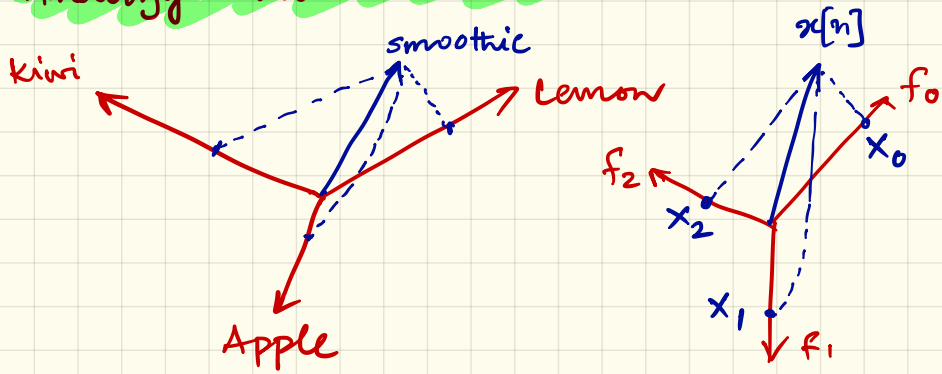
$$x[n] = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} X(m) e^{j\frac{2\pi}{N} \cdot mn}$$



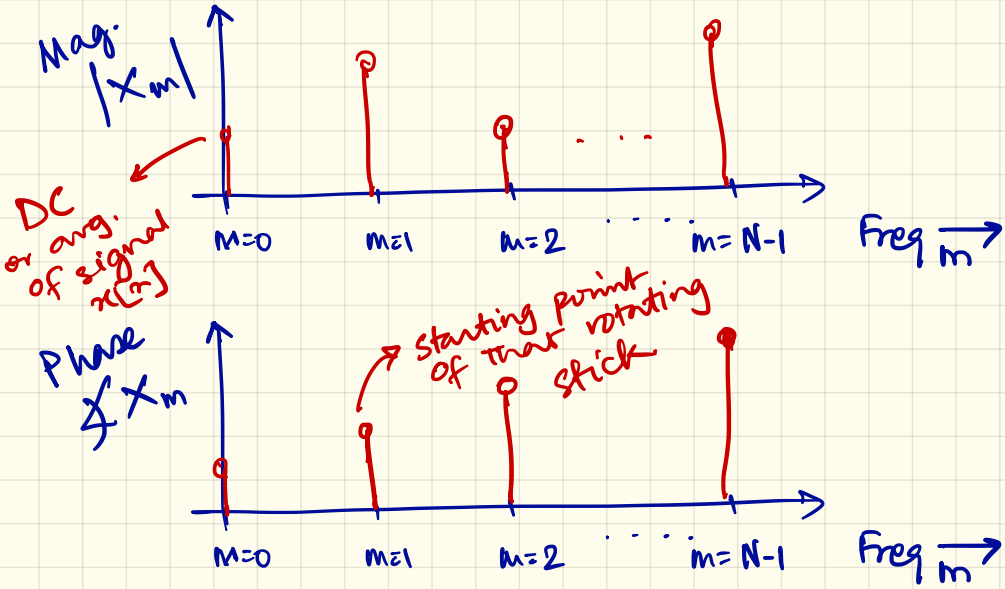
$\Rightarrow$ Add up the projections multiplied by the corresponding basis vector.

$$x[1] = X(1) e^{j\frac{2\pi}{N} \cdot 1 \cdot n} + X(2) e^{j\frac{2\pi}{N} \cdot 2n} + X(3) e^{j\frac{2\pi}{N} \cdot 3n}$$

Analogy with Smoothie :



∴ Of course $X(m)$ is complex # so how to plot the DFT ?



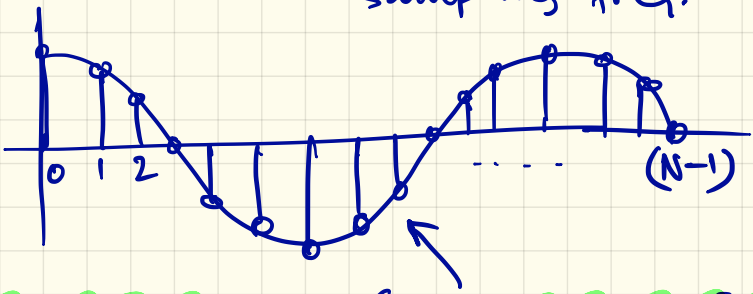
Max freq. of a given signal ⇒ BANDWIDTH

Examples and Properties

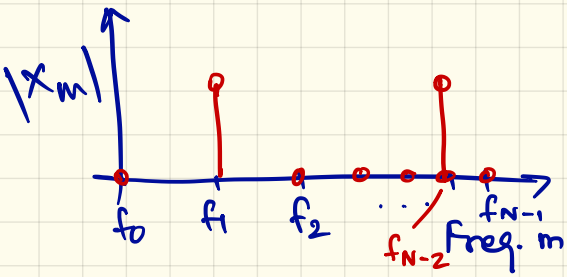
Everyday signal $\cos 2\pi f_m t$

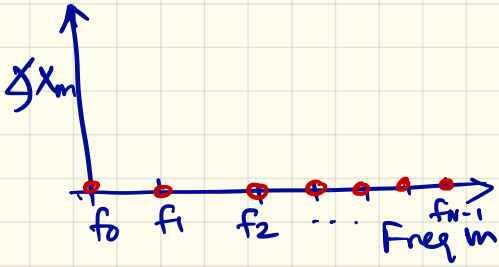
Discrete sampled signal $x[n] = \cos 2\pi f_m n t_s$

where $t_s = \frac{1}{f_s}$ is sampling interval, f_s is sampling freq.

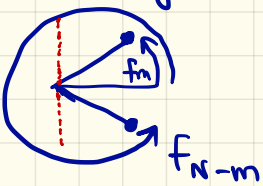


Now what is the DFT ($x[n] = \cos 2\pi f_1 n t_s$)



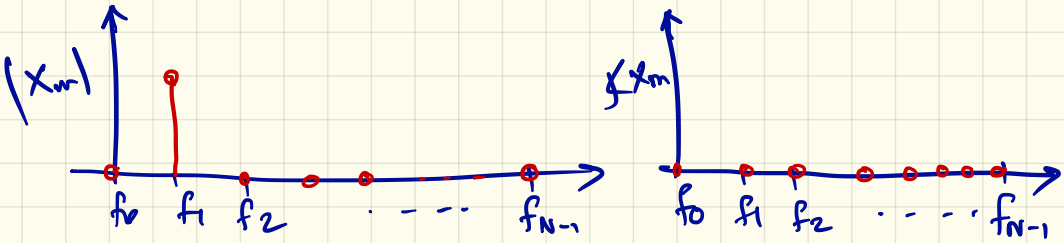


Real signals always symmetric because the imaginary component needs to be cancelled out by a much faster moving rotation



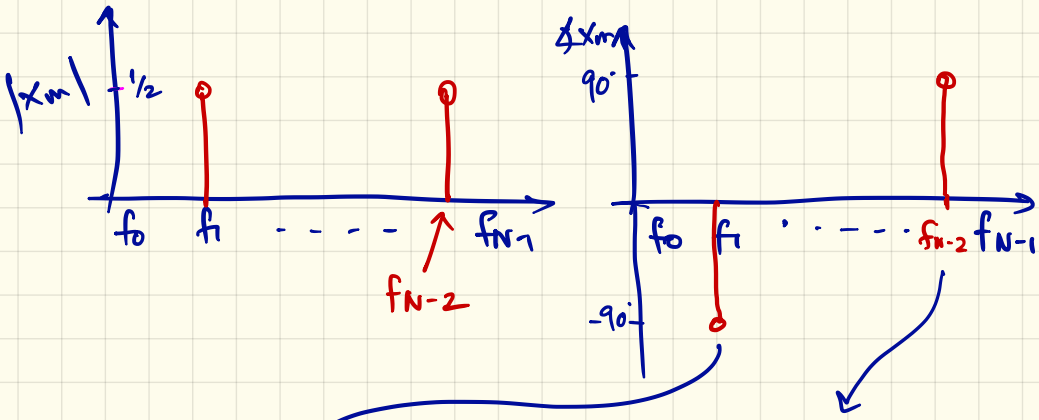
Also, phase ZERO because all rotations start from zero.

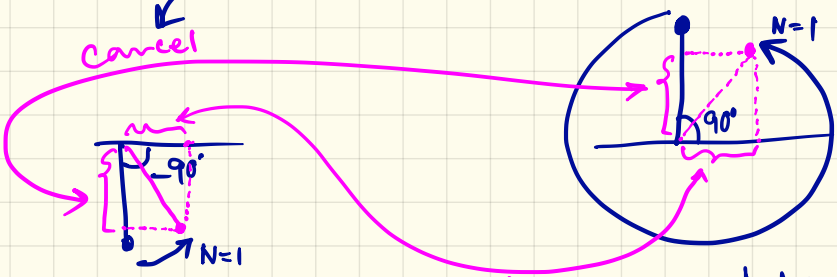
(→) What is DFT($x[n] = e^{j2\pi f_1 n t_s}$) ?



Just one stick rotating at f_1 is good enough to create this signal.

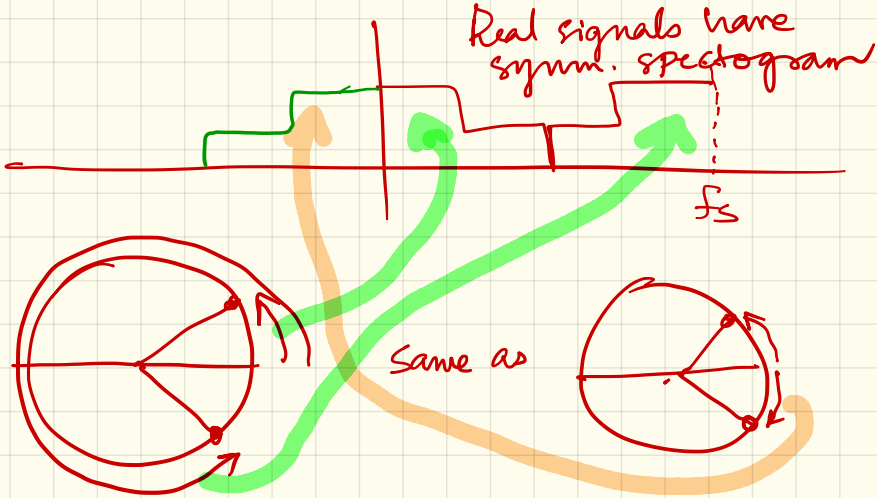
(→) Now, what is DFT($x[n] = \sin 2\pi f_1 n t_s$) ?





add up to 2x but multiplied by $x_m = 1/2$

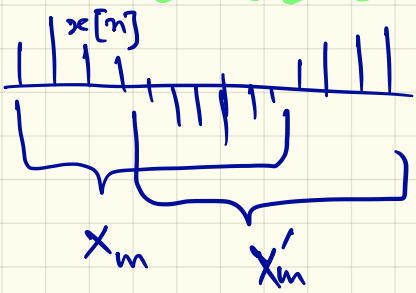
Symmetric and Negative Freq:



So this is equivalent to taking the mirror image and pasting it on the (-)ve freq. axis.

Intuitively, the second stick is rotating in the clockwise direction to cancel the imaginary parts of the real signal.

DFT of a shifted signal?

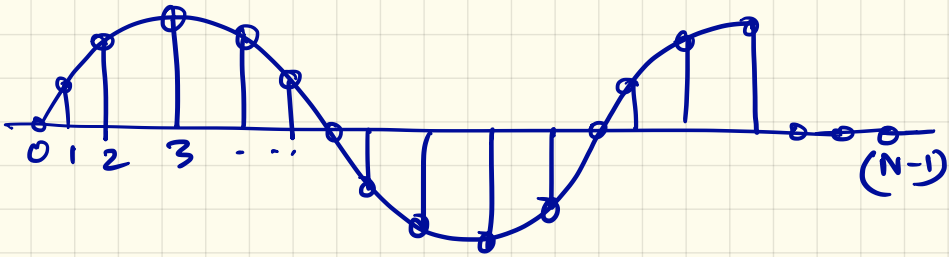


$$x'[n] = x[n+k]$$

i.e., $x'[n]$ is shifted by k samples. Then

$$X'_m = \underbrace{e^{j\frac{2\pi}{N}\cdot k\cdot m}}_{\text{means only phase shift}} X_m$$

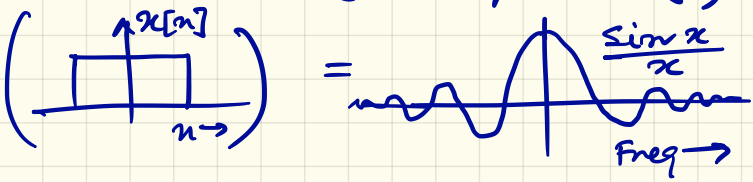
- Nyquist sampling theorem, $f_s \geq 2W$
- Relation between DFT, f_s, N.
- Practical cases where N does not cover integer cycles of each freq. component.



$\Downarrow$

Basic idea of DFT (Rectangular sig.)

DFT



N samples in 1 second

f_s samples in 1 second

1 sample in $\frac{1}{f_s}$ second

$\therefore$ N samples in $\frac{N}{f_s}$ seconds

$\therefore$ freq. $= \frac{f_s}{N}$