Model multiplicity → count/numbers.

Model opposite → I have $5
               How much should I add to make $0?  Ans: -5

Numbers & neg. numbers are all abstractions of reality.

Now, how to model rotation?

We have Cosθ and Sinθ.
Of course we need to express their relation in some way. We cannot just say
\[ \text{Cos}θ + \text{Sin}θ \times \]

or \[ \langle \text{Cos}θ, \text{Sin}θ \rangle \]
Which still does not allow algebra.

OK, we can say \[ \hat{x} \text{Cos}θ + \hat{y} \text{Sin}θ \]
But we don’t have to say \[ \hat{x} \], we just need to make \[ \hat{y} \] become 1 to \[ \hat{z} \].

How? Well, one way is to say that \[ \hat{y} \] should be \[ -1\hat{x} \].
i.e., \[ y^2 = -1 \]  \[ \therefore \ y = \sqrt{-1} = j \]

\[ \therefore \text{ Rotation is a vector of } \text{Cos}θ + j\text{Sin}θ \]
Aha, this is exactly \[ e^{jθ} \] models rotation.
Now, what would rotation at different freq. look like? Say we want to model every freq. in N discrete steps. Fournier observed each column in orthogonal:

\[ f_0 \cdot f_1 = 0 \Rightarrow \text{you can immediately see because} \]

Same holds for any pair.

\[
\begin{bmatrix}
\varepsilon^{0} & \varepsilon^{0} & \varepsilon^{0} & \cdots & \varepsilon^{0} \\
\varepsilon^{0} & \varepsilon^{0} & \varepsilon^{1} & \cdots & \varepsilon^{1} \\
\varepsilon^{0} & \varepsilon^{1} & \varepsilon^{2} & \cdots & \varepsilon^{2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\varepsilon^{0} & \varepsilon^{N-1} & \varepsilon^{N-1} & \cdots & \varepsilon^{N-1}
\end{bmatrix}
\]
Fourier Matrices → Signal Processing

→ Negative nos. don't exist in reality; it's an abstraction.
  
  Negative IR 0 positive real line
  is imagination that helps math.

→ Complex # (a+jb) also such an abstraction.
  - Physical interpretation of j is 90° rotation.
  - So (a+jb) is a on Real axis and b after 90° rotation.
  - This is why (j²) is (-1), (j³) = -i which is 90° in the opposite direction.

→ Now consider a stick rotating.

→ Now, what is e^{j\theta}?
  - It's = \cos \theta + j \sin \theta.
  - As \theta changes, we get a circle.

→ Enumerate the values of e^{j\theta} (spinning stick) at different points during spin.
  
  Say spinning we enumerate at granularity of
  \frac{2\pi}{N} slices.

  \[
  \begin{bmatrix}
  e^{j0} \\
  e^{j2\pi/N} \\
  e^{j4\pi/N} \\
  \vdots \\
  e^{j(N-1)2\pi/N}
  \end{bmatrix}
  \]

  Let's spin a little faster, say double.

  \[
  \begin{bmatrix}
  e^{j0} \\
  e^{j2\pi/N} \\
  e^{j4\pi/N} \\
  \vdots \\
  e^{j(N-1)2\pi/N}
  \end{bmatrix}
  \]
In general: 

\[ e^{i \left( \frac{2\pi}{N} \right) n} \]

\[
\begin{bmatrix}
  e^0 & e^0 & e^0 \\
  e^{2\pi/N} & e^{2\pi/N} & e^{(N-1)2\pi/N} \\
  e^{2\pi/N} & e^{2(2\pi/N)} & e^{2(N-1)2\pi/N} \\
  \vdots & \vdots & \vdots \\
  e^{(N-1)2\pi/N} & e^{(N-1)(2\pi/N)} & e^{(N-1)(N-1)2\pi/N} \\
\end{bmatrix}
\]

\[
\begin{align*}
  m &= 1 \\
  m &= 2 \\
  m &= (N-1)
\end{align*}
\]

This matrix is \(N \times N\).

AND, any column of this matrix is any other column.

\[ \rightarrow \text{Note: each column corresponds to an increasing freq. of movement/spinning.} \]

This implies, that the different frequencies form a basis of the space of frequencies.

\[ \rightarrow \text{i.e., each column is a vector denoting freq. } m. \]

Now, when you have data, say GPS location \( <\text{lat, long, elevation}> \)

You can project on any orthogonal basis, and the projected values (when summed) completely capture all info. in that data.

\[ \text{i.e., } GPS \rightarrow [x] + GPS [y] + GPS [z] \]

So long as \(x, y, z\) are bases.
Now any data can be treated as vectors.

By Parseval's analysis identity:

\[ X(m) = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi m n}{N}} \]

Projection of the \( X \) vector to the \( m^{th} \) column of the matrix (or \( m^{th} \) frequency).

Fourier Transform of \( X(t) \) or \( X[n] \).

- IDFT
- cos + sin signal
- cos 2\pi ft signal
- Mag + phase
- Time shift = phase shift
- Read fs \& freq.
Thus DFT is a change of basis:

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

This basis in the I matrix

This is the Fourier basis matrix F

\[
\begin{bmatrix}
1 & \ldots & 1 \\
\vdots & \ddots & \vdots \\
1 & \ldots & 1
\end{bmatrix}
\]

Note that dot product = Projection only when the dot product is normalized.

Now, for DFT, observe that each column of F is not a unit vector, e.g., \([1 \ 1 \ 1]^T\).

The DFT needs to be scaled by the length of the vector \(1/\sqrt{N}\)

\[
\text{DFT } X_m = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n] e^{-j \frac{2\pi}{N} mn}
\]

\[
\text{IDFT } x[n] = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} X_m e^{j \frac{2\pi}{N} mn}
\]
**IDFT:** Basically apply Parseval's synthesis identity:

\[
x[n] = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} X(m) e^{j \frac{2\pi}{N} - mn}
\]

Add up the projections multiplied by the corresponding basis vector.

\[
x[1] = X(1) e^{j \frac{2\pi}{N} \cdot 1 \cdot n} + X(2) e^{j \frac{2\pi}{N} \cdot 2 \cdot n} + X(3) e^{j \frac{2\pi}{N} \cdot 3 \cdot n}
\]
Analogy with smoothie:

- Kiwi
- Lemon
- Apple

Of course \( X(m) \) is complex # so how to plot the DFT?

- DC or avg. of signal \( x[n] \)
- Phase \( X_m \)
- Starting point of that rotating stick

Max freq. of a given signal \( \Rightarrow \) BANDWIDTH
Examples and Properties

Everyday signal $\cos 2\pi f m$

Discrete sampled signal $x[n] = \cos 2\pi f_m n t_s$

where $t_s = \frac{1}{f_s}$ is sampling interval, $f_s$ is sampling freq.

Now what is the DFT ($x[n] = \cos 2\pi f m n t_s$)

Real signals always symmetric because the imaginary component needs to be cancelled out by a much faster moving rotation.

Also, phase ZERO because all rotations start from zero.
What is $DFT(\alpha[n] = e^{j2\pi f_1 n})$?

Just one stick rotating at $f_1$ is good enough to create this signal.

Now, what is $DFT(\alpha[n] = \sin 2\pi f_1 n)$?

Add up to $\alpha[n]$ but multiplied by $\alpha[n] = \frac{1}{2}$
Symmetric and Negative Freq.

Real signals have
symm. spectrum

Real signals have
symm. spectrum

So this is equivalent to taking the mirror image and pasting it on the negative freq.
axis.

Intuitively, the second stick is rotating in the clockwise direction to cancel the imaginary parts of the real signal.

DFT of a shifted signal?

$x'[n] = x[n+k]$ 
i.e., $x'[n]$ is shifted by $k$ samples. Then

$x'_m = e^{j \frac{2\pi}{N} k m} x_m$

means only phase shift
The Nyquist sampling theorem states that for a signal to be perfectly reconstructed, the sampling rate must be at least twice the highest frequency component of the signal, i.e., \( f_s \geq 2f_H \).

The relation between the Discrete Fourier Transform (DFT), the sampling rate \( f_s \), and the number of samples \( N \) is crucial. In practical cases where \( N \) does not cover integer cycles of each frequency component, a basic idea of the DFT can be visualized:

\[
\text{DFT} \left( \underbrace{\text{Rectangular sig.}}_{\text{signal}} \right) = \underbrace{\sin \frac{x}{x}}_{\text{frequency}} \quad \text{Freq} \rightarrow
\]

- \( N \) samples in 1 second
- \( f_s \) samples in 1 second
- 1 sample in \( \frac{1}{fs} \) second

\[ \therefore N \text{ samples in } \frac{N}{fs} \text{ seconds} \]

\[ \therefore \text{freq.} = \frac{fs}{2} \]