ECE 418: Introduction to Image & Video Processing

Multidimensional Signal Processing

Shuai Huang

01/15/2019

University of Illinois at Urbana-Champaign
1. Introduction

2. Two-dimensional signal processing
   2.1 Two-dimensional Fourier transform
   2.2 Properties of DSFT
   2.3 DSFT examples
   2.4 2D impulse distribution
Introduction
Introduction: image processing applications

Remote sensing by satellites and spacecrafts.

Figure 1: Various platforms and sensors used for remote sensing [1].
Introduction: image processing applications

Remote sensing by satellites and spacecrafts.

Figure 2: Automating the Detection of Landslides\(^1\).

\(^1\)https://earthobservatory.nasa.gov/images/88319/automating-the-detection-of-landslides
Introduction: image processing applications

Medical image processing: computed tomography (CT), magnetic resonance imaging (MRI), etc.

Figure 3: Left: The MRI machine; Right: The MRI of a knee².

Introduction: image processing applications

Super-resolution imaging.

Figure 4: Simple magnification compared with SR³.

³https://www.extremetech.com/extreme/132950-csi-style-super-resolution-image-enlargment-yeeaaah
Introduction: video processing applications

Ultra-high-definition TV: 4K UHD, 8K UHD.

Figure 5: Comparison of 8K UHDTV (7680 × 4320), 4K UHDTV (3840 × 2160), HDTV (1920 × 1080) and SDTV (720 × 480) resolutions [2].
Introduction: video processing applications

Online streaming services: YouTube, Netflix, HBO, etc.

Figure 6: Various online streaming services⁴.

⁴https://www.wired.com/2015/03/ways-new-apple-tv-dominate-living-room/
Introduction: what is an image?

- Light intensity (pictures from ordinary camera).

Figure 7: An image taken with a cell phone camera.
Introduction: what is an image?

- Absorption characteristics of medium (X-ray imaging).

Figure 8: Wilhelm Rontgen took this radiograph of his wife’s left hand on December 22, 1895, shortly after his discovery of X-rays.

NATIONAL LIBRARY OF MEDICINE
Introduction: what is an image?

- Temperature profile of a region (infrared imaging)

Figure 9: Thermogram of a cat [3].
Introduction

What is an image?
Any function $f(x, y)$ of spatial coordinates $(x, y) \in \mathbb{R}^2$ that conveys information is an image.

What is a video?
An image that evolves in time $(t) \rightarrow$ a 3D function $f(x, y, t)$. 
Introduction

Digital image processing:

- Image $f(x, y)$ is discretized → array of samples (pixels)
- Pixel values are quantized.
- Process sampled, quantized image.

Figure 10: $f(x, y)$ is discretized to pixels, then quantized to integers $\in [0, 255]$. 
Digital image processing:

- Image $f(x, y)$ is **discretized** → array of samples (pixels)
- Pixel values are **quantized**.
- Process sampled, quantized image.

Digital video processing:

- Video $f(x, y, t)$ is sampled, quantized and processed.
Introduction: advantages in digital processing

Why digital image & video processing?

- flexibility, affordability, portability.
- stored “indefinitely” without error and degradation.
- easy duplication, transmission.
- post-processing with special purpose hardware and software/algorithms.

And many more ...
Introduction: challenges in digital processing

2-D (3-D) versus 1-D?

- much larger data size.
- similar operations (sampling, filtering, transformation), yet harder mathematics.
- spacial domain \((x, y)\) and time domain \(t\) treated \textbf{differently}.
- more degrees of freedom: sampling pattern versus sampling interval.
Two-dimensional signal processing
2D signal processing

Images are represented as 2-D discrete signals with finite support. Typical images are

- in rectangular shape.
- approximated by equally spaced samples on a rectangular grid.

Figure 11: 2-D signal with rectangular support.
2D signal processing

Let $\vec{n} = [n_1, n_2]$ denote a vector in $\mathbb{Z}^2$

$$x(\vec{n}) = \begin{cases} 
  x(n_1, n_2) & \text{if } \vec{n} \in [0, N_1 - 1] \times [0, N_2 - 1] \\
  0 & \text{otherwise}
\end{cases}$$  \hspace{1cm} (1)

Figure 11: 2-D signal with rectangular support.
Special 2D signals

Signal symmetric around 0

\[ x(-\vec{n}) = x(\vec{n}) \]  

Circular symmetric signal

\[ x(\vec{n}) \] is a function of \( \|\vec{n}\| \) only.

\[ x(\vec{n}) = f(\|\vec{n}\|) = f(n_1^2 + n_2^2) \]
Special 2D signals

Periodic signal $\tilde{x}$

\[
\begin{align*}
\tilde{x}(\vec{n} + \vec{N}_1) &= \tilde{x}(\vec{n}) \\
\tilde{x}(\vec{n} + \vec{N}_2) &= \tilde{x}(\vec{n})
\end{align*}
\tag{4}
\]

$\vec{N}_1, \vec{N}_2 \in \mathbb{Z}^2$ are linearly independent with integer entries.

Figure 12: Periodic signal with periodicity vectors $\vec{N}_1$ and $\vec{N}_2$. 21
Special 2D signals

Separable signal

\[ x(n_1, n_2) = x_1(n_1) \cdot x_2(n_2) \] \hspace{1cm} (5)

2D unit impulse

The signal is 1 at the origin and 0 elsewhere

\[ x(\vec{n}) = \begin{cases} 
1 & \text{if } n_1 = n_2 = 0 \\
0 & \text{otherwise}
\end{cases} \] \hspace{1cm} (6)
2D Fourier Transform of discrete signals

Discrete-space Fourier transform (DSFT):

\[ X_d(\omega_1, \omega_2) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} x(n_1, n_2) e^{-j(\omega_1 n_1 + \omega_2 n_2)} \]  

- From discrete spatial domain \((n_1, n_2)\) to continuous frequency domain \((\omega_1, \omega_2)\).
- \(n_1, n_2\) are integers \(\mathbb{Z}\).
- \(\omega_1, \omega_2\) are real numbers \(\mathbb{R}\).

\[ X_d(\omega) = \sum_{\vec{n}} x(\vec{n}) e^{-j \omega^T \vec{n}} \]  

(8)
2D Fourier Transform of discrete signals

Inverse DSFT:

\[
x(n_1, n_2) = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} X_d(\omega_1, \omega_2)e^{j(\omega_1 n_1 + \omega_2 n_2)} \, d\omega_1 \, d\omega_2
\]  

(9)

○ From continuous frequency domain \((\omega_1, \omega_2)\) to discrete spatial domain \((n_1, n_2)\).

\[
x(\vec{n}) = \frac{1}{(2\pi)^2} \int_{[-\pi, \pi] \times [-\pi, \pi]} X_d(\vec{\omega})e^{j\vec{\omega}^T \vec{n}} \, d\vec{\omega}
\]  

(10)
Properties of DSFT

Existence

\[ X_d(\omega_1, \omega_2) \text{ exists if } \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} |x(n_1, n_2)| < \infty. \]

Periodicity in the frequency domain

\[
X_d \left( \vec{\omega} + \begin{bmatrix} 2\pi \\ 0 \end{bmatrix} \right) = X_d(\vec{\omega}) \quad (11a)
\]

\[
X_d \left( \vec{\omega} + \begin{bmatrix} 0 \\ 2\pi \end{bmatrix} \right) = X_d(\vec{\omega}) \quad (11b)
\]
Properties of DSFT

Linearity

\[ z(\vec{n}) = \alpha x(\vec{n}) + \beta y(\vec{n}) \]  
\[ Z_d(\vec{\omega}) = \alpha X_d(\vec{\omega}) + \beta Y_d(\vec{\omega}) \]

(12a)  
(12b)

Shifting

\[ y(\vec{n}) = x(\vec{n} + \vec{k}) \]  
\[ Y_d(\vec{\omega}) = e^{j\vec{\omega}^T \vec{k}} X_d(\vec{\omega}) \]

(13a)  
(13b)
Properties of DSFT

Parseval’s theorem

\[
\sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} |x(n_1, n_2)|^2 = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |X_d(\omega_1, \omega_2)|^2 \ d\omega_1 d\omega_2
\]

(14)

- The energy of the signal \( x \):
  \[\sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} |x(n_1, n_2)|^2\]

- The spectral power of \( x \):
  \[\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |X_d(\omega_1, \omega_2)|^2 \ d\omega_1 d\omega_2\]
Properties of DSFT

For real signal \( x(\hat{n}) \in \mathbb{R}^2 \),

\[
\begin{align*}
\text{Re}[X_d(-\hat{\omega})] &= \text{Re}[X_d(\hat{\omega})] \quad (15) \\
\text{Im}[X_d(-\hat{\omega})] &= -\text{Im}[X_d(\hat{\omega})] \quad (16)
\end{align*}
\]

For real signal \( x(\hat{n}) \in \mathbb{R}^2 \),

\[
\begin{align*}
|X_d(-\hat{\omega})| &= |X_d(\hat{\omega})| \quad (17) \\
\angle X_d(-\hat{\omega}) &= -\angle X_d(\hat{\omega}) \quad (18)
\end{align*}
\]
Properties of DSFT

For real, symmetric signal \( x(\vec{n}) \in \mathbb{R}^2 \) and \( x(-\vec{n}) = x(\vec{n}) \),

\[
\text{Im}[X_d(\vec{\omega})] = 0 \quad (19)
\]

- The frequency of a real, symmetric signal only has real component.

For separable signal \( x(n_1, n_2) = x_1(n_1) \cdot x_2(n_2) \),

\[
X_d(\vec{\omega}) = \text{DTFT}[x_1](\omega_1) \cdot \text{DTFT}[x_2](\omega_2) \quad (20)
\]

- The DSFT of a separable signal is also separable.
Consider the signal shown in Fig. 13,

\[ x(\vec{n}) = \begin{cases} 
1 & : \|\vec{n}\| \leq 1 \\
0 & : \text{else} 
\end{cases} \]  

(21)

\begin{itemize}
  \item The signal is circularly symmetric and nonseparable.
\end{itemize}

Figure 13: A circular symmetric, nonseparable signal.
Example 1 of DSFT

DSFT of a symmetric signal is real-valued.

\[
X_d(\vec{\omega}) = \sum_{n_1 = -\infty}^{\infty} \sum_{n_2 = -\infty}^{\infty} x(n_1, n_2) e^{-j(\omega_1 n_1 + \omega_2 n_2)}
\]

\[
= x(0, 0) + x(1, 0) e^{-j\omega_1} + x(-1, 0) e^{j\omega_1}
\]

\[
+ x(0, 1) e^{-j\omega_2} + x(0, -1) e^{j\omega_2}
\]

\[
= 1 + 2 \cos(\omega_1) + 2 \cos(\omega_2)
\]
Example 2 of DSFT

Consider the separable signal $x(n_1, n_2) = x_1(n_1) \cdot x_2(n_2)$ shown in Fig. 14,

$$x_1(k) = x_2(k) = \begin{cases} 
0.5 & : k = 0 \\
0.25 & : |k| = 1 \\
0 & : \text{else}
\end{cases} \quad (23)$$

Figure 14: A symmetric, separable signal.
Example 2 of DSFT

The DSFT of a separable signal is also separable:

\[
\text{DTFT}[x_1](\omega_1) = \sum_{k=-\infty}^{\infty} x_1(k)e^{-j\omega_1 k}
\]

\[
= x_1(-1)e^{j\omega_1} + x_1(0) + x_1(1)e^{-j\omega_1}
\]

\[
= 0.5(1 + \cos(\omega_1))
\]

\[
\text{DTFT}[x_2](\omega_2) = 0.5(1 + \cos(\omega_2))
\]

\[
X_d(\hat{\omega}) = \text{DTFT}[x_1](\omega_1) \cdot \text{DTFT}[x_2](\omega_2)
\]

\[
= 0.25(1 + \cos(\omega_1))(1 + \cos(\omega_2))
\]
Example 3 of DSFT

Consider the windowed 2D sinusoid signal in Fig. 17,

\[
x(\vec{n}) = \begin{cases} 
\cos(\lambda_1 n_1 + \lambda_2 n_2) & : 0 \leq n_1 < N_1, 0 \leq n_2 < N_2 \\
0 & : \text{else}
\end{cases}
\]

(27)

Figure 15: A real, nonsymmetric and nonseparable signal: 
\(\lambda_1 = \lambda_2 = 0.01, N_1 = N_2 = 1000.\)
Example 3 of DSFT

Do it in the hard way with the following formula:

From Euler’s formula:  \( \cos(\phi) = e^{j\phi} + e^{-j\phi} \)  

Geometric series:  \[ \sum_{n=0}^{N-1} x^{an} = \frac{1 - x^{aN}}{1 - x^a}, \quad x \neq 1 \]  

Figure 16: DSFT of a windowed 2D sinusoid: \( \lambda_1 = \lambda_2 = 0.01, N_1 = N_2 = 1000. \)
2D impulse distribution

2D impulse distribution $\delta(\cdot)$

$\delta(\cdot)$ maps a whole function $f(\cdot, \cdot)$ to the single number $f(0, 0)$:

$$f(t_1, t_2) \rightarrow f(0, 0)$$

$$\delta[f(t_1, t_2)] := f(0, 0) \quad (30)$$

$\delta(\cdot)$ is just a mathematical abstraction, often visualized inaccurately as follows:

Figure 17: 2D Dirac impulse with unit volume.
2D impulse distribution

A popular notation for 2D impulse distribution:

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(t_1, t_2)f(t_1, t_2) dt_1 dt_2 = f(0, 0) \tag{31}
\]

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(t_1, t_2) dt_1 dt_2 = 1 \tag{32}
\]

◎ \(\delta(t_1, t_2)\) in (31) is not a function.

◎ The integral sign \(\int\) in (31) is not an integral.

◎ (31) is just an notation for (30).
2D impulse distribution

Why isn’t $\delta(t_1, t_2)$ a function?

- A function should have definite value at each point in its domain.
- The domain of $\delta(t_1, t_2)$ is a set containing all the possible function values.

$$S = \{s_{t_1,t_2} | s_{t_1,t_2} = f(t_1, t_2), -\infty < t_1, t_2 < \infty\} \quad (33)$$

- The set $S$ changes if $f(t_1, t_2)$ changes. The value $f(0, 0)$ also changes if $f(t_1, t_2)$ changes.
- The mapping from $S$ to $f(0, 0)$ is not definite.
Why isn’t the integral sign $\int$ in $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(t_1, t_2) dt_1 dt_2 = 1$ an integral?

○ Because $\delta(t_1, t_2)$ is not a function. The integration is inoperable.
○ It is just an notation.
Acknowledgment: The slides are based on the lecture notes by Prof. Pierre Moulin, UIUC.

