UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN
Department of Electrical and Computer Engineering

ECE 417 Principles of Signal Analysis
Spring 2014

EXAM 3 SOLUTIONS

Friday, May 9, 2014

- This is a CLOSED BOOK exam.
- There are a total of 100 points in the exam. Each problem specifies its point total. Plan your work accordingly.
- You must SHOW YOUR WORK to get full credit.

Problem	Score
1	
2	
3	
4	
5	
6	
Total	

Name: ______________________________

Useful Angles

θ	$\cos\theta$	$\sin\theta$	$e^{j\theta}$
0	1	0	1
$\pi/6$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}/2 + j/2$
$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	$\sqrt{2}/2 + j\sqrt{2}/2$
$\pi/3$	$1/2$	$\sqrt{3}/2$	$1/2 + j\sqrt{3}/2$
$\pi/2$	0	1	j
π	-1	0	-1
$3\pi/2$	1	-1	$-j$
2π	1	0	1

Gaussian Probability Densities (to Two Significant Figures)

x	$\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$
0	0.40
0.5	0.35
1	0.24
1.5	0.13
2	0.05
2.5	0.02
3	0.00

Other Possibly Useful Formulas

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

$$h[n] = \frac{\sin\omega_c n}{\pi n} \leftrightarrow H(e^{j\omega}) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & \text{otherwise} \end{cases}$$

$$u[n] - u[n-N] \leftrightarrow e^{-j\frac{\omega(N-1)}{2}}\frac{\sin(\omega N/2)}{\sin(\omega/2)}$$

$$\delta[n] \leftrightarrow 1$$

$$e^{j\alpha n} \leftrightarrow 2\pi\delta(\omega - \alpha)$$

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}$$

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j2\pi kn/N}$$

$$S = \sum_{k=1}^{n}(\vec{x}_k - \vec{m})(\vec{x}_k - \vec{m})^T$$

Problem 1 (21 points)

You are given a 640x480 B/W input image, $x[n_1, n_2]$ for integer pixel values $0 \leq n_1 \leq 639$, $0 \leq n_2 \leq 479$. You wish to interpolate the given pixel values in order to find the value of the image at location $(500.3, 300.8)$. Specify the formula used to calculate $x[500.3, 300.8]$ using each of the following algorithms. Be certain that your formula clearly states which pixels from the input image are used.

(a) Piece-wise constant interpolation.

$$x[500.3, 300.8] = x[500, 301]$$

(b) Bilinear interpolation.

$$\begin{aligned} x[500.3, 300.8] &= (0.7)(0.2)x[500, 300] + (0.7)(0.8)x[500, 301] + \\ &\quad (0.3)(0.2)x[501, 300] + (0.3)(0.8)x[501, 301] \end{aligned}$$

(c) Sinc interpolation.

$$x[500.3, 300.8] = \sum_{n_1=0}^{639} \sum_{n_2=0}^{479} x[n_1, n_2] \mathrm{sinc}\left(\pi(500.3 - n_1)\right) \mathrm{sinc}\left(\pi(300.8 - n_2)\right)$$

Problem 2 (24 points)

The images $y[\vec{\eta}]$ and $x[\vec{m}]$ are related by an affine transformation, where $\vec{\eta} = [\eta, \xi, 1]^T$ and $\vec{m} = [m, n, 1]$ are coordinate vectors of the input and output image, respectively, m is the row index, and n is the column index.

(a) The affine transformation $\vec{\eta} = A\vec{m}$ is a rotation by $-\frac{\pi}{3}$ radians. Find A.

$$A = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) The affine transformation $\vec{\eta} = B\vec{m}$ consists of scaling the height of the image (m) by a factor of 5, while keeping the width (n) unchanged. Find B.

$$B = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(c) The affine transformation $\vec{\eta} = C\vec{m}$ consists of shifting all pixels to the left (negative n direction) by 20 columns. Find C.

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -20 \\ 0 & 0 & 1 \end{bmatrix}$$

(d) The affine transformation $\vec{\eta} = D\vec{m}$ consists of performing parts (a) through (c) of this problem, one after the other, in order. Specify the matrix D in terms of the matrices A, B, and C. **There should be no numbers in your answer to this part**.

$$D = CAB$$

Problem 3 (11 points)

A particular triangle has corner coordinates at

$$\vec{x}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \vec{x}_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Let $\vec{\lambda}_0 = [\lambda_1, \lambda_2, \lambda_3]^T$ be the barycentric coordinate vector corresponding to pixel $\vec{x}_0 = \left[\frac{2}{3}, \frac{1}{3}\right]^T$. Find $\vec{\lambda}_0$.

$$\vec{\lambda}_0 = \begin{bmatrix} 0 \\ 1/3 \\ 2/3 \end{bmatrix}$$

Problem 4 (12 points)

The images $y[\vec{\eta}]$ and $x[\vec{m}]$ are related by an affine transformation $\vec{\eta} = A\vec{m}$, where $\vec{\eta} = [\eta, \xi, 1]^T$ and $\vec{m} = [m, n, 1]^T$ are coordinate vectors of the input and output images, respectively. It is known that under this transformation, the origin swaps places with the point $[2, 2]$, thus

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 2 \\ 2 \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Specify the A matrix as completely as you can. There should be two scalar variables in your answer; you may use the variables names α and β.

$$A = \begin{bmatrix} \alpha & -1-\alpha & 2 \\ \beta & -1-\beta & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Problem 5 (16 points)

You are creating a recommender system that tries to recommend songs that will be considered to be similar to a given query. Each song is characterized by a two-dimensional vector $\vec{x}_k = [b_k, v_k]^T$ where b_k is the number of beats per minute, and v_k is the fraction of air-time during which there is a human voice. Your customer considers the following four songs to be similar:

$$[\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4] = \begin{bmatrix} 120 & 140 & 140 & 120 \\ 0.3 & 0.3 & 0.5 & 0.5 \end{bmatrix}$$

You are given two more test data, $\vec{x}_5 = [b_5, v_5]^T$ and $\vec{x}_6 = [b_6, v_6]^T$, and you are asked whether or not $\vec{x}_5$ and $\vec{x}_6$ should be considered similar. Write formulas for the Mahalanobis distance between $\vec{x}_5$ and $\vec{x}_6$ under the following conditions:

(a) Estimate a diagonal data covariance matrix directly from the data, and use it to write the squared Mahalanobis distance $d^2_\Sigma(\vec{x}_5, \vec{x}_6)$.

$$d^2_\Sigma(\vec{x}_5, \vec{x}_6) = \frac{(b_5 - b_6)^2}{100} + \frac{(v_5 - v_6)^2}{0.01}$$

(b) Estimate a diagonal data covariance matrix from the data, then regularize it using regularization parameter $\lambda = 0.01$ before using the result to write the squared Mahalanobis distance $d^2_\Sigma(\vec{x}_5, \vec{x}_6)$.

$$d^2_\Sigma(\vec{x}_5, \vec{x}_6) = \frac{(b_5 - b_6)^2}{100.01} + \frac{(v_5 - v_6)^2}{0.02}$$

Problem 6 (16 points)

A particular 6 megapixel image contains 3 million red pixels ($[r, g, b] = [255, 0, 0]$) and 3 million blue pixels ($[r, g, b] = [0, 0, 255]$).

Define its 8-quantile color histogram $h[k_R, k_G]$ to be an 8×8 table of numbers, specifying the number of pixels having redshift in the k_R^{th} quantile (where smaller k_R indicates smaller redshift, $0 \leq k_R \leq 7$), and greenshift in the k_G^{th} quantile ($0 \leq k_G \leq 7$).

(a) Find $h[k_R, k_G]$.

$$h[k_R, k_G] = \begin{cases} 3 \times 10^6 & k_R = 7, k_G = 0 \\ 3 \times 10^6 & k_R = 0, k_G = 0 \\ 0 & \text{otherwise} \end{cases}$$

(b) Suppose that there is another 6 megapixel image with 3 million black pixels ($[r, g, b] = [1, 1, 1]$) and 3 million white pixels ($[r, g, b] = [255, 255, 255]$). Say that the color histogram of this image is called $g[k_R, k_G]$. What is $\|g[k_R, k_G] - h[k_R, k_G]\|$, the distance between the color histogram of the black-white image and the color histogram of the red-blue image?

$$g[k_R, k_G] = \begin{cases} 6 \times 10^6 & k_R = 2, k_G = 2 \\ 0 & \text{otherwise} \end{cases}$$

Correct answer can be either the ℓ_1 or ℓ_2 norm:

$$\|g[k_R, k_G] - h[k_R, k_G]\|_2 = 3\sqrt{6} \times 10^6$$

$$\|g[k_R, k_G] - h[k_R, k_G]\|_1 = 12 \times 10^6$$