Lecture 6: Optical Flow

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Outline

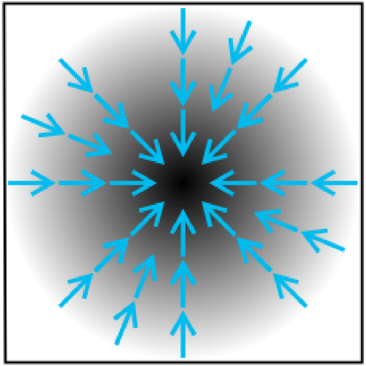
1. Image Gradient
2. Optical Flow
3. The Lucas-Kanade Algorithm
4. Pseudo-Inverse
5. Summary

Image gradient

The image gradient is a way of characterizing the distribution of light and dark pixels in an image. Suppose the image intensity is $f(t, r, c)$. The image gradient is:

$$\nabla f == \left[\begin{array}{c} \frac{\partial f(t,r,c)}{\partial c} \\ \frac{\partial f(t,r,c)}{\partial r} \end{array}\right]$$

Image gradient



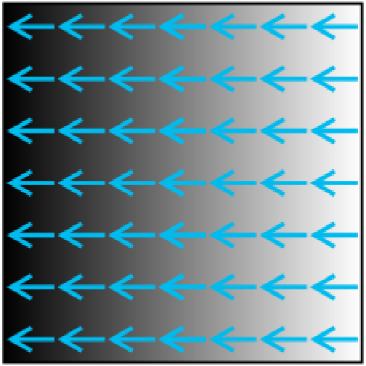
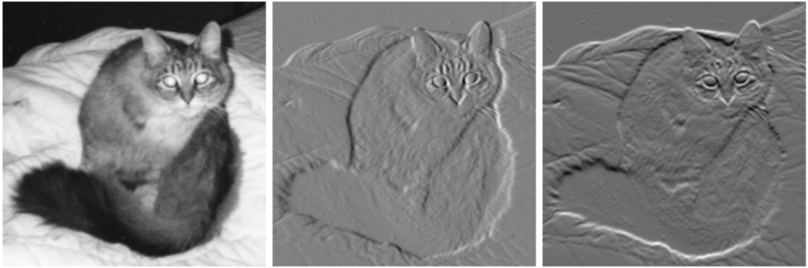


Image gradient



Public domain image, Njw00, 2010

How do you calculate the image gradient?

Basically, use one of the standard numerical estimates of a derivative. For example, the central-difference operator:

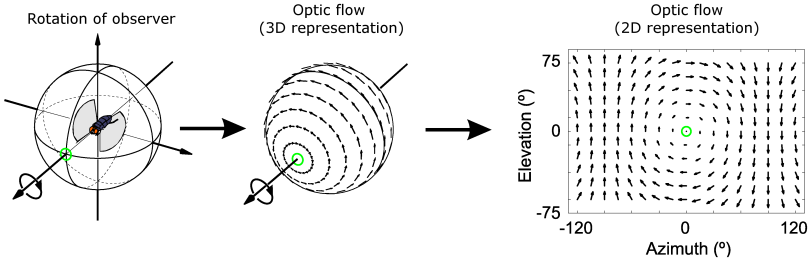
$$\nabla f = \begin{bmatrix} \frac{\partial f(t,r,c)}{\partial c} \\ \frac{\partial f(t,r,c)}{\partial r} \end{bmatrix} = \begin{bmatrix} \frac{f[t,r,c+1]-f[t,r,c-1]}{2} \\ \frac{f[t,r+1,c]-f[t,r-1,c]}{2} \end{bmatrix}$$

Wikipedia has a good listing of other methods you can use.

Outline

Optical Flow

Definition: **optical flow** is the vector field $\vec{v}(t, r, c)$ specifying the current apparent velocity of the pixel at position (r, c). It depends on motion of (1) the object observed, and (2) the observer.



Optical Flow

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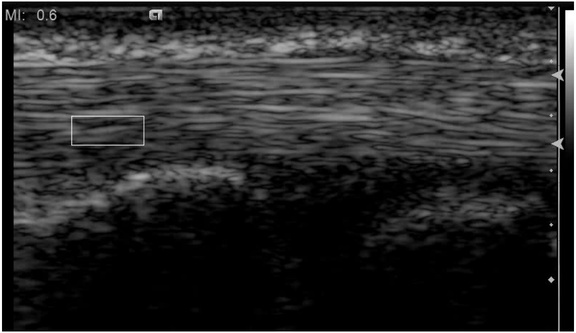
Pengcheng Han et. al. "An Object Detection Method Using Wavelet Optical Flow and Hybrid Linear-Nonlinear Classifier", Mathematical Problems in Engineering doi:10.1155/2013/96541

Optical Flow

For example, you can use it to track a user-specified rectangle in the ultrasound video of a tendon.



 Chuang B, Hsu J, Kuo L, Jou I, Su F, Sun Y (2017). "Tendon-motion tracking in an ultrasound image sequence using optical-flow-based block matching". BioMedical Engineering OnLine

How to calculate optical flow

General idea:

- Treat the image as a function of continuous time and space, $f(t, r, c)$.
- If the image intensity is changing, as a function of time, then try to explain it by moving pixels around.

Calculating optical flow

More formally, let's treat local variation of $f(t,r,c)$ using a first-order Taylor series:

$$f\left(t+\Delta t, r+\Delta r, c+\Delta c\right) \approx f(t,r,c) + \Delta t\frac{\partial f}{\partial t} + \Delta r\frac{\partial f}{\partial r} + \Delta c\frac{\partial f}{\partial c}$$

Hypothesize that all intensity variations are caused by pixels moving around. Then

$$f\left(t+\Delta t, r+\Delta r, c+\Delta c\right) - f(t,r,c) = 0$$

Calculating optical flow

$$
\begin{aligned}
0 &= f\left(t+\Delta t, r+\Delta r, c+\Delta c\right) - f(t,r,c) \\
&\approx \Delta t \frac{\partial f}{\partial t} + \Delta r \frac{\partial f}{\partial r} + \Delta c \frac{\partial f}{\partial c}
\end{aligned}
$$

Calculating optical flow

$$0 \approx \Delta t \frac{\partial f}{\partial t} + \Delta r \frac{\partial f}{\partial r} + \Delta c \frac{\partial f}{\partial c}$$

Dividing through by Δt, and taking the limit as $\Delta t \rightarrow 0$, we get

$$0 \approx \frac{\partial f}{\partial t} + \left(\frac{\partial r}{\partial t}\right) \frac{\partial f}{\partial r} + \left(\frac{\partial c}{\partial t}\right) \frac{\partial f}{\partial c}$$

Calculating optical flow

Re-arranging gives us the optical flow equation:

$$-\frac{\partial f}{\partial t} \approx \left(\frac{\partial r}{\partial t}\right)\frac{\partial f}{\partial r} + \left(\frac{\partial c}{\partial t}\right)\frac{\partial f}{\partial c}$$

How to calculate optical flow

Define the optical flow vector, $\vec{v}(t, r, c)$, and image gradient, $\nabla f(t, r, c)$:

$$\vec{v} = \begin{bmatrix} \frac{\partial c}{\partial t} \\ \frac{\partial r}{\partial t} \end{bmatrix}, \quad \nabla f = \begin{bmatrix} \frac{\partial f}{\partial c} \\ \frac{\partial f}{\partial r} \end{bmatrix}$$

Then the optical flow equation is:

$$-\frac{\partial f}{\partial t} = (\nabla f)^T \vec{v}$$

Outline

How to calculate optical flow

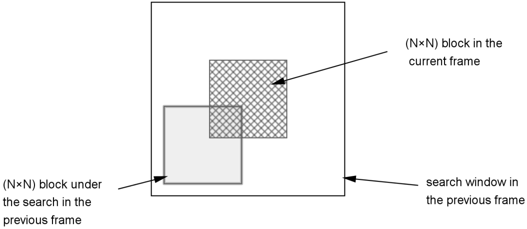
So we have this optical flow equation:

$$-\frac{\partial f}{\partial t} = (\nabla f)^T \vec{v}$$

Assume that we can calculate $\partial f / \partial t$ and ∇f, using standard image gradient methods. Now we just need to find $\vec{v}$. But $\vec{v} = [v_r, v_c]^T$ is a vector of two unknowns, so the equation above is one equation in two unknowns!

How to calculate optical flow

The solution is to assume that a small block of pixels all move together:



The Lucas-Kanade Algorithm

The Lucas-Kanade Algorithm replaces this equation

$$-\frac{\partial f}{\partial t} = (\nabla f)^T \vec{v}$$

with this equation:

$$-\left[\begin{array}{c} \frac{\partial f[t,r-W,c-W]}{\partial t} \\ \vdots \\ \frac{\partial f[t,r+W,c+W]}{\partial t} \end{array}\right] = \left[\begin{array}{cc} \frac{\partial f[t,r-W,c-W]}{\partial c} & \frac{\partial f[t,r-W,c-W]}{\partial r} \\ \vdots & \\ \frac{\partial f[t,r+W,c+W]}{\partial c} & \frac{\partial f[t,r+W,c+W]}{\partial r} \end{array}\right] \left[\begin{array}{c} v_c \\ v_r \end{array}\right]$$

so that we are averaging over a block of size $(2W+1) \times (2W+1)$ pixels.

The Lucas-Kanade Algorithm

The Lucas-Kanade algorithm solves the equation

$$\vec{b} = A\vec{v}$$

where

$$\vec{b} = -\left[\begin{array}{c} \frac{\partial f[t,r-W,c-W]}{\partial t} \\ \vdots \\ \frac{\partial f[t,r+W,c+W]}{\partial t} \end{array}\right], \quad A = \left[\begin{array}{cc} \frac{\partial f[t,r-W,c-W]}{\partial c} & \frac{\partial f[t,r-W,c-W]}{\partial r} \\ \vdots & \\ \frac{\partial f[t,r+W,c+W]}{\partial c} & \frac{\partial f[t,r+W,c+W]}{\partial r} \end{array}\right]$$

$$\vec{v} = \left[\begin{array}{c} v_c[t,r,c] \\ v_r[t,r,c] \end{array}\right]$$

The Lucas-Kanade Algorithm

The Lucas-Kanade algorithm solves the equation

$$\vec{b} = A\vec{v}$$

...but now A is a matrix of size $(2W+1)\times 2$, so it's still not invertible! How do we solve that?

Outline

Pseudo-Inverse

The pseudo-inverse, $A^\dagger$, of any matrix A, is a matrix that acts like A^{-1} in many ways, but it doesn't require A to be square. Here are some of its properties:

$$AA^\dagger A = A$$
$$A^\dagger A A^\dagger = A$$

Pseudo-Inverse

Of particular interest to us, the vector $\vec{v} = A^\dagger \vec{b}$ "pseudo-solves" the equation $\vec{b} = A\vec{v}$. By pseudo-solve, we mean that

- If A is a short fat matrix, then there are an infinite number of different vectors $\vec{v}$ that solve $\vec{b} = A\vec{v}$. $\vec{v} = A^\dagger \vec{b}$ is one of those; specifically, it's the one that minimizes $\|\vec{v}\|^2$.
- If A is a tall thin matrix, then there is usually no vector $\vec{v}$ that solves $\vec{b} = A\vec{v}$, but $\vec{v} = A^\dagger \vec{b}$ is the vector that comes closest, in the sense that

$$A^\dagger \vec{b} = \text{argmin}_{\vec{v}} \|\vec{b} - A\vec{v}\|^2$$

Solving for the Pseudo-Inverse

Let's use this equation:

$$v^* = A^\dagger \vec{b} = \text{argmin}_v \|\vec{b} - A\vec{v}\|^2$$

to solve for the pseudo-inverse.

Solving for the Pseudo-Inverse

$$
\begin{aligned}
A^{\dagger}\vec{b} &= \text{argmin}_v \|\vec{b} - A\vec{v}\|^2 \\
&= \text{argmin}_v \left(\vec{b} - A\vec{v}\right)^T \left(\vec{b} - A\vec{v}\right) \\
&= \text{argmin}_v \left(\vec{b}^T\vec{b} - 2\vec{v}^T A^T\vec{b} + \vec{v}^T A^T A\vec{v}\right)
\end{aligned}
$$

Solving for the Pseudo-Inverse

$$A^\dagger \vec{b} = \text{argmin}_v \left(\vec{b}^T \vec{b} - 2\vec{v}^T A^T \vec{b} + \vec{v}^T A^T A \vec{v} \right)$$

If we differentiate the quantity in parentheses, and set the derivative to zero, we get

$$\vec{0} = -2A^T \vec{b} + 2A^T A \vec{v}$$

Assume that the columns of A are linearly independent; then $A^T A$ is invertible, and so the solution is

$$\vec{v} = (A^T A)^{-1} A^T \vec{b}$$

Outline

Summary: Optical Flow

- Optical flow is the vector field, $\vec{v}(t,r,c)$, as a function of pixel position and frame number.
- It is computed by assuming that the only changes to an image are the ones caused by motion, so that

$$f(t+\Delta t, r+\Delta r, c+\Delta c) = f(t,r,c)$$

- From that assumption, we get the optical flow equation:

$$-\frac{\partial f}{\partial t} = \vec{v}^T \nabla f$$

The Lucas-Kanade Algorithm

Lucas-Kanade assumes that there is a $(2W+1)\times(2W+1)$ block of pixels that all move together, so that $\vec{b}=A\vec{v}$, where

$$\vec{b}=-\left[\begin{array}{c}\frac{\partial f[t,r-W,c-W]}{\partial t}\\ \vdots\\ \frac{\partial f[t,r+W,c+W]}{\partial t}\end{array}\right],\quad A=\left[\begin{array}{cc}\frac{\partial f[t,r-W,c-W]}{\partial c} & \frac{\partial f[t,r-W,c-W]}{\partial r}\\ \vdots & \\ \frac{\partial f[t,r+W,c+W]}{\partial c} & \frac{\partial f[t,r+W,c+W]}{\partial r}\end{array}\right]$$

$$\vec{v}=\left[\begin{array}{c}v_c[t,r,c]\\ v_r[t,r,c]\end{array}\right]$$

Pseudo-Inverse

The Lucas-Kanade equation cannot be solved exactly, because it is $(2W+1)$ equations in only two unknowns. But we can find the minimum-squared error solution, which is

$$v^*[t,r,c] = A^\dagger \vec{b} = (A^T A)^{-1} A^T \vec{b}$$