Lecture 6: Optical Flow

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Outline

Image gradient

The image gradient is a way of characterizing the distribution of light and dark pixels in an image. Suppose the image intensity is $f(t,r,c)$. The image gradient is:

$$\nabla f == \left[\begin{array}{c} \frac{\partial f(t,r,c)}{\partial c} \\ \frac{\partial f(t,r,c)}{\partial r} \end{array}\right]$$

Image gradient

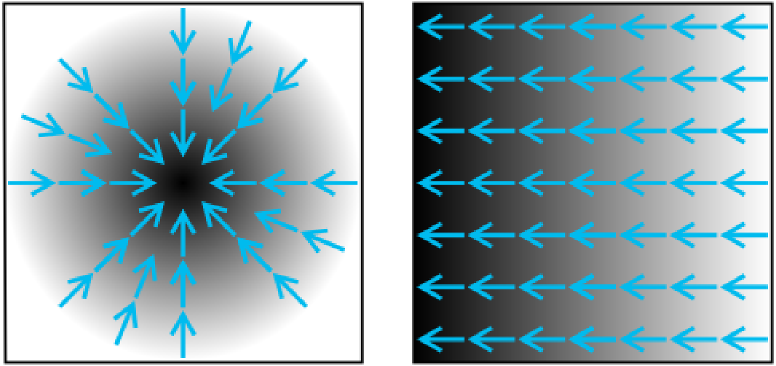
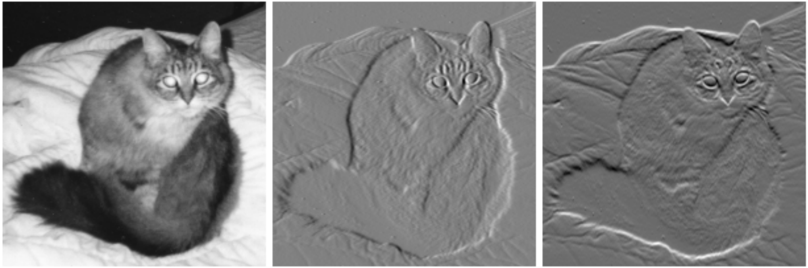


Image gradient



Public domain image, Njw00, 2010

How do you calculate the image gradient?

Basically, use one of the standard numerical estimates of a derivative. For example, the central-difference operator:

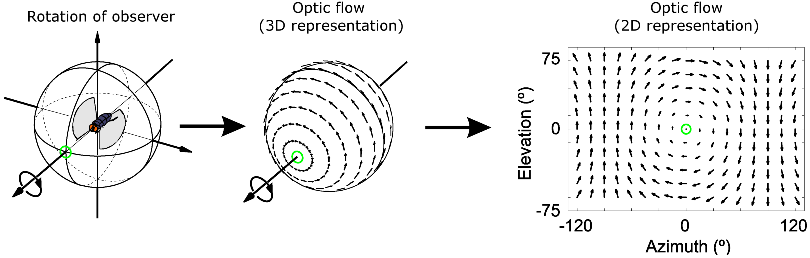
$$\nabla f = \begin{bmatrix} \frac{\partial f(t,r,c)}{\partial c} \\ \frac{\partial f(t,r,c)}{\partial r} \end{bmatrix} = \begin{bmatrix} \frac{f[t,r,c+1]-f[t,r,c-1]}{2} \\ \frac{f[t,r+1,c]-f[t,r-1,c]}{2} \end{bmatrix}$$

Wikipedia has a good listing of other methods you can use.

Outline

Optical Flow

Definition: **optical flow** is the vector field $\vec{v}(t, r, c)$ specifying the current apparent velocity of the pixel at position (r, c). It depends on motion of (1) the object observed, and (2) the observer.



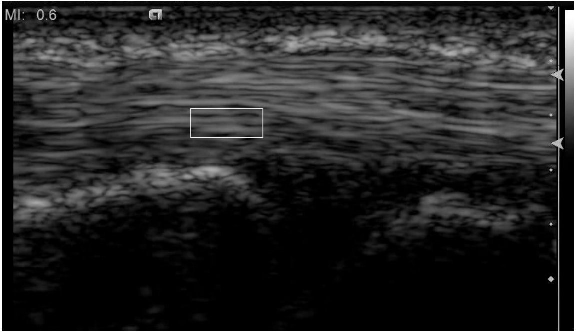
Optical Flow

Definition: **optical flow** is the vector field $\vec{v}(t, r, c)$ specifying the current apparent velocity of the pixel at position (r, c). It depends on motion of (1) the object observed, and (2) the observer.

Pengcheng Han et. al. "An Object Detection Method Using Wavelet Optical Flow and Hybrid Linear-Nonlinear Classifier", Mathematical Problems in Engineering doi:10.1155/2013/96541

Optical Flow

For example, you can use it to track a user-specified rectangle in the ultrasound video of a tendon.



CC-BY 4.0 Chuang B, Hsu J, Kuo L, Jou I, Su F, Sun Y (2017). "Tendon-motion tracking in an ultrasound image sequence using optical-flow-based block matching". BioMedical Engineering OnLine

How to calculate optical flow

General idea:

- Treat the image as a function of continuous time and space, $f(t, r, c)$.
- If the image intensity is changing, as a function of time, then try to explain it by moving pixels around.

Calculating optical flow

More formally, let's treat local variation of $f(t, r, c)$ using a first-order Taylor series:

$$f(t + \Delta t, r + \Delta r, c + \Delta c) \approx f(t, r, c) + \Delta t \frac{\partial f}{\partial t} + \Delta r \frac{\partial f}{\partial r} + \Delta c \frac{\partial f}{\partial c}$$

Hypothesize that all intensity variations are caused by pixels moving around. Then

$$f(t + \Delta t, r + \Delta r, c + \Delta c) - f(t, r, c) = 0$$

Calculating optical flow

$$\begin{aligned}
0 &= f\left(t+\Delta t, r+\Delta r, c+\Delta c\right) - f(t,r,c) \\
&\approx \Delta t \frac{\partial f}{\partial t} + \Delta r \frac{\partial f}{\partial r} + \Delta c \frac{\partial f}{\partial c}
\end{aligned}$$

Calculating optical flow

$$0 \approx \Delta t \frac{\partial f}{\partial t} + \Delta r \frac{\partial f}{\partial r} + \Delta c \frac{\partial f}{\partial c}$$

Dividing through by Δt, and taking the limit as $\Delta t \rightarrow 0$, we get

$$0 \approx \frac{\partial f}{\partial t} + \left(\frac{\partial r}{\partial t}\right) \frac{\partial f}{\partial r} + \left(\frac{\partial c}{\partial t}\right) \frac{\partial f}{\partial c}$$

Calculating optical flow

Re-arranging gives us the optical flow equation:

$$-\frac{\partial f}{\partial t} \approx \left(\frac{\partial r}{\partial t}\right)\frac{\partial f}{\partial r} + \left(\frac{\partial c}{\partial t}\right)\frac{\partial f}{\partial c}$$

How to calculate optical flow

Define the optical flow vector, $\vec{v}(t,r,c)$, and image gradient, $\nabla f(t,r,c)$:

$$\vec{v} = \left[\begin{array}{c} \frac{\partial c}{\partial t} \\ \frac{\partial r}{\partial t} \end{array}\right], \quad \nabla f = \left[\begin{array}{c} \frac{\partial f}{\partial c} \\ \frac{\partial f}{\partial r} \end{array}\right]$$

Then the optical flow equation is:

$$-\frac{\partial f}{\partial t} = (\nabla f)^T \vec{v}$$

Outline

How to calculate optical flow

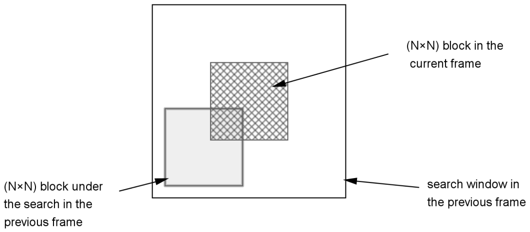
So we have this optical flow equation:

$$-\frac{\partial f}{\partial t} = (\nabla f)^T \vec{v}$$

Assume that we can calculate $\partial f / \partial t$ and ∇f, using standard image gradient methods. Now we just need to find $\vec{v}$. But $\vec{v} = [v_r, v_c]^T$ is a vector of two unknowns, so the equation above is one equation in two unknowns!

How to calculate optical flow

The solution is to assume that a small block of pixels all move together:



The Lucas-Kanade Algorithm

The Lucas-Kanade Algorithm replaces this equation

$$-\frac{\partial f}{\partial t} = (\nabla f)^T \vec{v}$$

with this equation:

$$-\left[\begin{array}{c} \frac{\partial f[t,r-W,c-W]}{\partial t} \\ \vdots \\ \frac{\partial f[t,r+W,c+W]}{\partial t} \end{array}\right] = \left[\begin{array}{cc} \frac{\partial f[t,r-W,c-W]}{\partial c} & \frac{\partial f[t,r-W,c-W]}{\partial r} \\ \vdots & \\ \frac{\partial f[t,r+W,c+W]}{\partial c} & \frac{\partial f[t,r+W,c+W]}{\partial r} \end{array}\right] \left[\begin{array}{c} v_c \\ v_r \end{array}\right]$$

so that we are averaging over a block of size $(2W+1) \times (2W+1)$ pixels.

The Lucas-Kanade Algorithm

The Lucas-Kanade algorithm solves the equation

$$\vec{b} = A\vec{v}$$

where

$$\vec{b} = -\left[\begin{array}{c} \frac{\partial f[t,r-W,c-W]}{\partial t} \\ \vdots \\ \frac{\partial f[t,r+W,c+W]}{\partial t} \end{array}\right], \quad A = \left[\begin{array}{cc} \frac{\partial f[t,r-W,c-W]}{\partial c} & \frac{\partial f[t,r-W,c-W]}{\partial r} \\ \vdots & \\ \frac{\partial f[t,r+W,c+W]}{\partial c} & \frac{\partial f[t,r+W,c+W]}{\partial r} \end{array}\right]$$

$$\vec{v} = \left[\begin{array}{c} v_c[t,r,c] \\ v_r[t,r,c] \end{array}\right]$$

The Lucas-Kanade Algorithm

The Lucas-Kanade algorithm solves the equation

$$\vec{b} = A\vec{v}$$

...but now A is a matrix of size $(2W+1) \times 2$, so it's still not invertible! How do we solve that?

Outline

Pseudo-Inverse

The pseudo-inverse, $A^\dagger$, of any matrix A, is a matrix that acts like A^{-1} in many ways, but it doesn't require A to be square. Here are some of its properties:

$$AA^\dagger A = A$$
$$A^\dagger AA^\dagger = A$$

Pseudo-Inverse

Of particular interest to us, the vector $\vec{v} = A^\dagger \vec{b}$ "pseudo-solves" the equation $\vec{b} = A\vec{v}$. By pseudo-solve, we mean that

- If A is a short fat matrix, then there are an infinite number of different vectors $\vec{v}$ that solve $\vec{b} = A\vec{v}$. $\vec{v} = A^\dagger \vec{b}$ is one of those; specifically, it's the one that minimizes $\|\vec{v}\|^2$.
- If A is a tall thin matrix, then there is usually no vector $\vec{v}$ that solves $\vec{b} = A\vec{v}$, but $\vec{v} = A^\dagger \vec{b}$ is the vector that comes closest, in the sense that

$$A^\dagger \vec{b} = \text{argmin}_{\vec{v}} \|\vec{b} - A\vec{v}\|^2$$

Solving for the Pseudo-Inverse

Let's use this equation:

$$v^* = A^\dagger \vec{b} = \text{argmin}_v \|\vec{b} - A\vec{v}\|^2$$

to solve for the pseudo-inverse.

Solving for the Pseudo-Inverse

$$
\begin{aligned}
A^{\dagger}\vec{b} &= \text{argmin}_v \|\vec{b} - A\vec{v}\|^2 \\
&= \text{argmin}_v \left(\vec{b} - A\vec{v}\right)^T \left(\vec{b} - A\vec{v}\right) \\
&= \text{argmin}_v \left(\vec{b}^T\vec{b} - 2\vec{v}^T A^T \vec{b} + \vec{v}^T A^T A\vec{v}\right)
\end{aligned}
$$

Solving for the Pseudo-Inverse

$$A^\dagger \vec{b} = \mathrm{argmin}_v \left(\vec{b}^T \vec{b} - 2\vec{v}^T A^T \vec{b} + \vec{v}^T A^T A \vec{v} \right)$$

If we differentiate the quantity in parentheses, and set the derivative to zero, we get

$$\vec{0} = -2A^T \vec{b} + 2A^T A \vec{v}$$

Assume that the columns of A are linearly independent; then $A^T A$ is invertible, and so the solution is

$$\vec{v} = (A^T A)^{-1} A^T \vec{b}$$

Outline

Summary: Optical Flow

- Optical flow is the vector field, $\vec{v}(t,r,c)$, as a function of pixel position and frame number.
- It is computed by assuming that the only changes to an image are the ones caused by motion, so that

$$f(t+\Delta t, r+\Delta r, c+\Delta c) = f(t,r,c)$$

- From that assumption, we get the optical flow equation:

$$-\frac{\partial f}{\partial t} = \vec{v}^T \nabla f$$

The Lucas-Kanade Algorithm

Lucas-Kanade assumes that there is a $(2W+1)\times(2W+1)$ block of pixels that all move together, so that $\vec{b} = A\vec{v}$, where

$$\vec{b} = -\left[\begin{array}{c} \frac{\partial f[t,r-W,c-W]}{\partial t} \\ \vdots \\ \frac{\partial f[t,r+W,c+W]}{\partial t} \end{array}\right], \quad A = \left[\begin{array}{cc} \frac{\partial f[t,r-W,c-W]}{\partial c} & \frac{\partial f[t,r-W,c-W]}{\partial r} \\ \vdots & \\ \frac{\partial f[t,r+W,c+W]}{\partial c} & \frac{\partial f[t,r+W,c+W]}{\partial r} \end{array}\right]$$

$$\vec{v} = \left[\begin{array}{c} v_c[t,r,c] \\ v_r[t,r,c] \end{array}\right]$$

Pseudo-Inverse

The Lucas-Kanade equation cannot be solved exactly, because it is $(2W+1)$ equations in only two unknowns. But we can find the minimum-squared error solution, which is

$$v^*[t,r,c] = A^\dagger \vec{b} = (A^T A)^{-1} A^T \vec{b}$$