Problem 2.1

Let $A$ be a $2 \times 2$ matrix, and let $x$ be one of its elements. All of its other elements are known, and are given as:

$$A = \begin{bmatrix} x & 3 \\ -1 & 2 \end{bmatrix}$$

The eigenvalues of $A$ are given by

$$\lambda = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

for some particular values of $a$, $b$, and $c$. Find $a$, $b$, and $c$, in terms of $x$, such that Equation (2.1-2) gives the eigenvalues of $A$.

Solution:

$$a = 1$$
$$b = -(x + 2)$$
$$c = 2x + 3$$

Problem 2.2

Let $A$ be a $2 \times 2$ matrix, and let $x$ be one of its elements. All of its other elements are known, and are given as:

$$A = \begin{bmatrix} x & 3 \\ -1 & 2 \end{bmatrix}$$

Suppose that you are given one of its eigenvalues, $\lambda$, and you want to find the corresponding eigenvector. As you know, the scale of an eigenvector is arbitrary, so let’s arbitrarily set its first element to 1: $\vec{v} = [1, v_2]^T$. Solve for its second element, $v_2$, in terms of $\lambda$.

Solution: Setting $A\vec{v} = \lambda\vec{v}$ gives two equations in one unknown: $x + 3v_2 = \lambda$, and $-1 + 2v_2 = \lambda v$. These will give the same answer if $\lambda$ is an eigenvalue:

$$v_2 = \frac{\lambda - x}{3} = \frac{1}{2 - \lambda}$$
Problem 2.3

Suppose that \( A \) is a tall thin matrix (more rows than columns). Suppose that \( A^\dagger = (A^T A)^{-1} A^T \) is its pseudo-inverse, and that \( \vec{v}^\ast = A^\dagger \vec{b} \). Show that \( \vec{v}^\ast \) is the minimum-squared error solution to the equation \( A\vec{v} \approx \vec{b} \), i.e., show that \( \vec{v}^\ast \) minimizes \( E = \|A\vec{v} - \vec{b}\|_2^2 \).

Solution:

\[
E = \|A\vec{v} - \vec{b}\|_2^2 \\
= (A\vec{v} - \vec{b})^T (A\vec{v} - \vec{b}) \\
= \vec{v}^T A^T A\vec{v} - 2\vec{v}^T A^T \vec{b} + \vec{b}^T \vec{b}
\]

Differentiating w.r.t. \( \vec{v} \) gives

\[
\nabla_{\vec{v}} E = 2A^T A\vec{v} - 2A^T \vec{b}
\]

Setting this to zero gives \( \vec{v}^\ast = (A^T A)^{-1} A^T \vec{b} \).

Problem 2.4

Suppose that \( A \) is a short fat matrix (more columns than rows). Suppose that \( A^\dagger = A^T (A A^T)^{-1} \) is its pseudo-inverse, and that \( \vec{v}^\ast = A^\dagger \vec{b} \). Show that \( \vec{v}^\ast \) satisfies the equation \( A\vec{v}^\ast = \vec{b} \).

Solution:

\[
A\vec{v}^\ast = A A^T (A A^T)^{-1} \vec{b} \\
= \vec{b}
\]