PRACTICE EXAM 1

Exam will be Tuesday, September 28, 2021

- This will be a CLOSED BOOK exam.
- You will be permitted one sheet of handwritten notes, 8.5x11.
- Calculators and computers are not permitted.
- If you’re taking the exam online, you will need to have your webcam turned on. Your exam will appear on Gradescope at exactly 9:30am; you will need to photograph and upload your answers by exactly 11:00am.
- There will be a total of 100 points in the exam. Each problem specifies its point total. Plan your work accordingly.
- You must SHOW YOUR WORK to get full credit.
1. (16 points) A 200 \times 200 sunset image is bright on the bottom, and dark on top, thus the pixel in the \(i\)th row and \(j\)th column has intensity \(A[i, j] = 200 - i\). Suppose that this formula extends past the edge of the image, and also extends to non-integer pixel locations, so you can assume that it holds for every real-valued coordinate value \((i, j)\).

Suppose your goal is to recognize sunset images that have been taken by people laying upside-down on the beach. You decide to train the classifier by rotating the image \(A[i, j]\) to every possible angle, thus creating the training images

\[
B_k[i, j] = A[i \cos \theta_k - \sin \theta_k, i \sin \theta_k + \cos \theta_k], \quad \theta_k = \frac{2\pi k}{100}, \quad 0 \leq k \leq 99
\]

Your next step is to reshape each 200 \times 200 image \(B_k[i, j]\) into a vector of raw pixel intensities, \(\vec{x}_k\), then to compute the dataset mean, \(\vec{m} = \frac{1}{100} \sum_{k=0}^{99} \vec{x}_k\).

(a) What is the length of the vector \(\vec{m}\)?

(b) What is the numerical value of \(\vec{m}\)? Provide enough information to specify the value of every element of the vector.
2. (16 points) Suppose you have a 1000-sample audio waveform, \( x[n] \), such that \( x[n] \neq 0 \) for \( 0 \leq n \leq 999 \). You want to chop this waveform into 200-sample frames, with 10\% overlap between frames.
How many nonzero samples are there in the last frame?

3. (21 points) You are given a 640x480 B/W input image, \( x[n_1, n_2] \) for integer pixel values \( 0 \leq n_1 \leq 639, 0 \leq n_2 \leq 479 \). You wish to interpolate the given pixel values in order to find the value of the image at location \((500.3, 300.8)\). Specify the formula used to calculate \( x[500.3, 300.8] \) using each of the following algorithms. Be certain that your formula clearly states which pixels from the input image are used.

(a) Piece-wise constant interpolation.

(b) Bilinear interpolation.

(c) Sinc interpolation.
4. (10 points) Suppose a particular image has the following pixel values:
\[ a[0, 0] = 1, \ a[1, 0] = 0, \ a[0, 1] = 0, \ a[1, 1] = 0 \]
Use bilinear interpolation to estimate the value of the pixel \( a\left(\frac{1}{3}, \frac{1}{3}\right)\).

5. (20 points) Image warping has moved input pixel \( i(4.6, 8.2) \) to output pixel \( i'(15, 7) \). Input pixel \( i(4.6, 8.2) \) is unknown, but you know that \( i(4, 8) = a, \ i(4, 9) = b, \ i(5, 8) = c, \) and \( i(5, 9) = d \). Use bilinear interpolation to estimate \( i(4.6, 8.2) \) in terms of \( a, b, c, \) and \( d \).

6. (17 points) Your goal is to find a positive real number, \( a \), so that \( ax[n] \) is as similar as possible to \( y[n] \) in the sense that it minimizes the following error:
\[
\epsilon = \int_{-\pi}^{\pi} \left( |Y(e^{j\omega})| - a|X(e^{j\omega})| \right)^2 d\omega
\]
Find the value of \( a \) that minimizes \( \epsilon \), in terms of \( |X(e^{j\omega})| \) and \( |Y(e^{j\omega})| \).
7. (5 points) Consider the problem of upsampling, by a factor of 2, the infinite-sized image

\[ x[n_1, n_2] = \delta[n_1 - 5] = \begin{cases} 
1 & n_1 = 5 \\
0 & \text{otherwise}
\end{cases} \]

Suppose that the image is upsampled, then filtered, as

\[ y[n_1, n_2] = \begin{cases} 
 x[n_1/2, n_2/2] & n_1/2 \text{ and } n_2/2 \text{ both integers} \\
0 & \text{otherwise}
\end{cases} \quad z[n_1, n_2] = y[n_1, n_2] \ast h[n_1, n_2] \]

Let \( h[n_1, n_2] \) be the ideal anti-aliasing filter with frequency response

\[ H(\omega_1, \omega_2) = \begin{cases} 
1 & |\omega_1| < \frac{\pi}{2}, \ |\omega_2| < \frac{\pi}{2} \\
0 & \text{otherwise}
\end{cases} \]

Find \( z[n_1, n_2] \).
8. (10 points) Consider the signal \( x[n] = \beta^n u[n] \), where \( u[n] \) is the unit step function.

(a) Find the LPC coefficient, \( \alpha \), that minimizes \( \varepsilon \), where

\[
\varepsilon = \sum_{n=-\infty}^{\infty} e^2[n], \quad e[n] = x[n] - \alpha x[n-1]
\]

(b) Find the signal \( e[n] \) that results from your choice of \( \alpha \) in part (a).
9. (10 points) Consider the LPC synthesis filter \( s[n] = e[n] + \alpha s[n - 1] \).

(a) Under what condition on \( \alpha \) is the synthesis filter stable?

(b) Assume that the synthesis filter is stable. Suppose that \( e[n] \) is the pulse train \( e[n] = \sum_{p=\infty}^{-\infty} \delta[n - pP] \). As a function of \( \alpha \), \( P \), and \( \omega \), what is the DTFT \( S(e^{j\omega}) \)? You need not simplify, but your answer should contain no integrals or infinite sums.

10. (5 points) Consider the synthesis filter \( s[n] = e[n] + bs[n - 1] - \left( \frac{b}{2} \right)^2 s[n - 2] \). For what values of \( b \) is the synthesis filter stable?
11. (5 points) Suppose you have a 200 \times 200-pixel image that is just one white dot at pixel (45, 25), and all the other pixels are black:

\[
x[n_1, n_2] = \begin{cases} 
255 & n_1 = 45, \ n_2 = 25 \\
0 & \text{otherwise}, \ 0 \leq n_1 < 199, \ 0 \leq n_2 < 199 
\end{cases}
\]

This image is upsampled to size 400 \times 400, then filtered, as

\[
y[n_1, n_2] = \begin{cases} 
x[n_1/2, n_2/2] & n_1/2 \text{ and } n_2/2 \text{ both integers} \\
0 & \text{otherwise}
\end{cases}
\]

\[
z[n_1, n_2] = y[n_1, n_2] \ast h[n_1, n_2]
\]

where \( h[n_1, n_2] \) is the ideal anti-aliasing filter whose frequency response is

\[
H(\omega_1, \omega_2) = \begin{cases} 
1 & |\omega_1| < \frac{\pi}{2}, \ |\omega_2| < \frac{\pi}{2} \\
0 & \text{otherwise}
\end{cases}
\]

Find \( z[n_1, n_2] \).
12. (10 points) Consider an infinite-sized grayscale image of a diagonal gray line:

\[ x[n_1, n_2] = \begin{cases} 
105 & n_1 - n_2 = 5 \\
0 & \text{otherwise} 
\end{cases} , \quad -\infty < n_1 < \infty, \quad -\infty < n_2 < \infty \]

(a) Suppose we **convolve each row** with a differencing filter:

\[ y[n_1, n_2] = x[n_1, n_2] * d_2[n_2], \quad d_2[n_2] = \begin{cases} 
1 & n_2 = 0 \\
-1 & n_2 = 2 \\
0 & \text{otherwise} 
\end{cases} \]

Find \( y[n_1, n_2] \).

(b) Suppose, INSTEAD, that we **convolve each row** with an averaging filter

\[ z[n_1, n_2] = x[n_1, n_2] * a_2[n_2], \quad a_2[n_2] = \begin{cases} 
1 & n_2 \in \{0, 2\} \\
2 & n_2 = 1 \\
0 & \text{otherwise} 
\end{cases} \]

Find \( z[n_1, n_2] \).
13. (15 points) A particular two-pole filter has the impulse response

\[ h[n] = e^{-\sigma_1 n} \sin(\omega_1 n) u[n] \]

\[ H(z) \] can be written as

\[ H(z) = \frac{G z^{-1}}{A(z)}, \quad A(z) = 1 - a_1 z^{-1} - a_2 z^{-2} \]

Find \( a_1, a_2, \) and \( G \) in terms of \( \sigma_1 \) and \( \omega_1 \).

14. (5 points) Suppose that \( \mathcal{X} \) is the unit disk, i.e.,

\[ \mathcal{X} = \left\{ \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : x_1^2 + x_2^2 \leq 1 \right\} \]

Suppose that \( \mathcal{Y} \) is defined as:

\[ \mathcal{Y} = \left\{ \vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} : \vec{y} = A \vec{x} \leq 1 \right\} \]

where \( A \) is defined to be the following matrix:

\[ A = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \]

Notice that the area of \( \mathcal{X} \), in the two-dimensional plane, is \( |\mathcal{X}| = \pi \). What is the numerical value of \( |\mathcal{Y}| \), the area of \( \mathcal{Y} \)?
15. (5 points) Suppose that you are trying to allocate money to a set of $N$ different possible investments. Suppose that if you allocate $a_k$ dollars to investment $k$, it will return $a_k b_k$ dollars in profit. However, in order to invest, you first need to borrow the money, and the cost of borrowing is quadratic in terms of the allocations. Let $\vec{a}$ be your vector of allocations, let $\vec{b}$ be the vector of profit factors, and let $C$ be the matrix of cost factors; suppose that your total profit is

$$P = \vec{b}^T \vec{a} - \vec{a}^T C \vec{a}$$

In terms of $\vec{b}$ and $C$, find the vector $\vec{a}$ that will maximize your profit. You may assume that $C$ is nonsingular.

16. (10 points) Suppose that you are watching a movie in which the camera is rotating around the top left corner of the frame at a rate of about 0.03 radians/frame, so that, as a function of the row index $r$ and column index $c$, the optical flow field is

$$\vec{v} = \begin{bmatrix} v_r \\ v_c \end{bmatrix} = \begin{bmatrix} 0.03c \\ -0.03r \end{bmatrix}$$

Suppose that the image $f[r,c]$ is a color gradient, with bright colors at the top of the image, and darker colors at the bottom:

$$f[r,c] = 255 - 0.1r$$

What is $\frac{\partial f}{\partial t}$, the change in pixel intensity, as a function of $r$ and $c$?