Lecture 1: Review of DTFT, Gaussians, and Linear Algebra

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ECE 417: Multimedia Signal Processing, Fall 2020

1. Outline of today's lecture
2. Review: DTFT
3. Review: Gaussians
4. Review: Linear Algebra
5. Summary

Outline

Outline of today's lecture

1. **Syllabus**
2. **Homework 1**
3. Review: DTFT, Gaussians, and Linear Algebra

What are the pre-requisites for ECE 417?

- ECE 310 **Digital Signal Processing**
- ECE 313 **Probability with Engineering Applications**
- Math 286 **Intro to Differential Eq Plus**

Outline

Discrete-Time Fourier Transform

The discrete-time Fourier transform of a signal $x[n]$ is

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

The inverse DTFT is

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$$

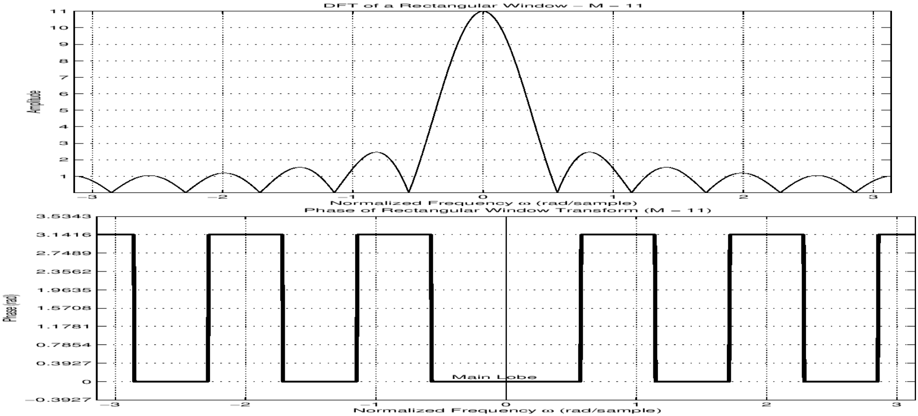
DTFT of a rectangle

One of the most important DTFTs you should know is the DTFT of a length-N rectangle:

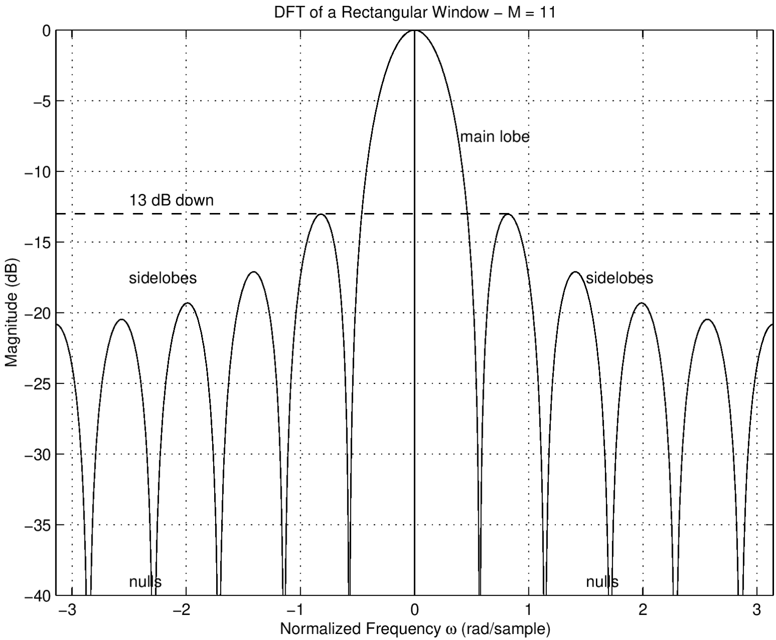
$$x[n] = u[n] - u[n-N] = \begin{cases} 1 & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$$

It is

$$X(\omega) = \sum_{n=0}^{N-1} e^{-j\omega n} = \frac{1 - e^{-j\omega N}}{1 - e^{-j\omega}} = e^{-j\omega\left(\frac{N-1}{2}\right)} \frac{\sin(\omega N/2)}{\sin(\omega/2)}$$

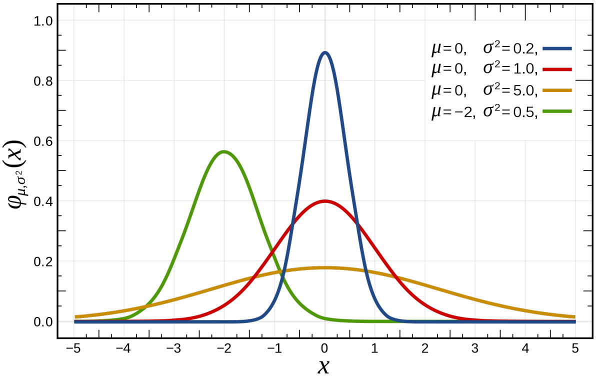


Smith, J.O. "The Rectangular Window", in Spectral Audio Signal Processing, **online book**, 2011 edition.


DFT of a Rectangular Window – M = 11
main lobe
13 dB down
sidelobes
sidelobes
nulls
nulls
Magnitude (dB)
Normalized Frequency ω (rad/sample)
0
−5
−10
−15
−20
−25
−30
−35
−40
−3
−2
−1
0
1
2
3

Outline

Gaussian (a.k.a. normal) pdf



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Normal pdf

A Gaussian random variable, X, is one whose probability density function is given by

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}$$

where μ and σ^2 are the mean and variance,

$$\mu = E\left[X\right], \quad \sigma^2 = E\left[(X-\mu)^2\right]$$

Standard normal

The cumulative distribution function (CDF) of a Gaussian RV is

$$F_X(x) = P\{X \le x\} = \int_{-\infty}^{x} f_X(y)dy = \int_{-\infty}^{(x-\mu)/\sigma} f_Z(y)dy = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

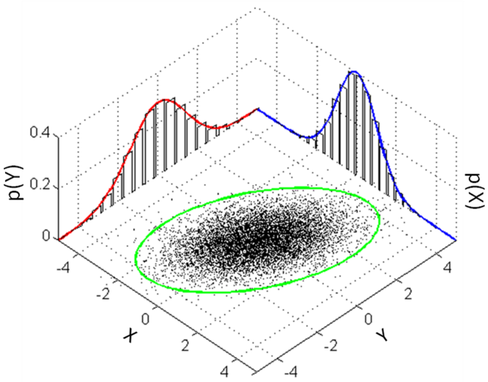
where $Z = \frac{X-\mu}{\sigma}$ is called the standard normal random variable. It is a Gaussian with zero mean, and unit variance:

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$$

We define $\Phi(z)$ to be the CDF of the standard normal RV:

$$\Phi(z) = \int_{-\infty}^{z} f_Z(y)dy$$

Multivariate normal pdf



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Jointly Gaussian Random Variables

Two random variables, X_1 and X_2, are jointly Gaussian if

$$f_{X_1,X_2}(x_1,x_2) = \frac{1}{2\pi|\Sigma|^{1/2}} e^{-\frac{1}{2}(\vec{x}-\vec{\mu})^T\Sigma^{-1}(\vec{x}-\vec{\mu})}$$

where $\vec{X}$ is the random vector, $\vec{\mu}$ is its mean, and Σ is its covariance matrix,

$$\vec{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, \quad \vec{\mu} = E\left[\vec{X}\right], \quad \Sigma = E\left[(\vec{X}-\vec{\mu})^T(\vec{X}-\vec{\mu})\right]$$

Covariance

The covariance matrix has four elements:

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \rho_{12} \\ \rho_{21} & \sigma_2^2 \end{bmatrix}$$

σ_1^2 and σ_2^2 are the variances of X_1 and X_2, respectively. $\rho_{12} = \rho_{21}$ is the covariance of X_1 and X_2:

$$\begin{aligned} \mu_1 &= E[X_1] \\ \sigma_1^2 &= E\left[(X_1 - \mu_1)^2\right] \\ \sigma_2^2 &= E\left[(X_2 - \mu_2)^2\right] \\ \rho_{12} &= E\left[(X_1 - \mu_1)(X_2 - \mu_2)\right] \end{aligned}$$

Jointly Gaussian Random Variables

$$f_{X_1,X_2}(x_1,x_2) = \frac{1}{2\pi|\Sigma|^{1/2}} e^{-\frac{1}{2}(\vec{x}-\vec{\mu})^T\Sigma^{-1}(\vec{x}-\vec{\mu})}$$

The multivariate normal pdf contains the determinant and the inverse of Σ. For a two-dimensional vector $\vec{X}$, these are

$$\Sigma = \left[\begin{array}{cc} \sigma_1^2 & \rho_{12} \\ \rho_{21} & \sigma_2^2 \end{array}\right]$$

$$|\Sigma| = \sigma_1^2\sigma_2^2 - \rho_{12}\rho_{21}$$

$$\Sigma^{-1} = \frac{1}{|\Sigma|}\left[\begin{array}{cc} \sigma_2^2 & -\rho_{12} \\ -\rho_{21} & \sigma_1^2 \end{array}\right]$$

Gaussian: Uncorrelated ⇔ Independent

Notice that if two Gaussian random variables are uncorrelated ($\rho_{12} = 0$), then they are also independent:

$$
\begin{aligned}
f_{X_1,X_2}(x_1,x_2) &= \frac{1}{2\pi|\Sigma|^{1/2}} e^{-\frac{1}{2}\frac{\left[\begin{array}{c} x_1-\mu_1 \\ x_2-\mu_2 \end{array}\right]^T \left[\begin{array}{cc} \sigma_2^2 & 0 \\ 0 & \sigma_1^2 \end{array}\right]\left[\begin{array}{c} x_1-\mu_1 \\ x_2-\mu_2 \end{array}\right]}{\sigma_1^2\sigma_2^2}} \\
&= \frac{1}{2\pi\sigma_1\sigma_2} e^{-\frac{1}{2}\left(\frac{(x_1-\mu_1)^2}{\sigma_1^2}+\frac{(x_2-\mu_2)^2}{\sigma_2^2}\right)} \\
&= \left(\frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{1}{2}\left(\frac{x_1-\mu_2}{\sigma_1}\right)^2}\right)\left(\frac{1}{\sqrt{2\pi\sigma_2^2}} e^{-\frac{1}{2}\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2}\right) \\
&= f_{X_1}(x_1) f_{X_2}(x_2)
\end{aligned}
$$

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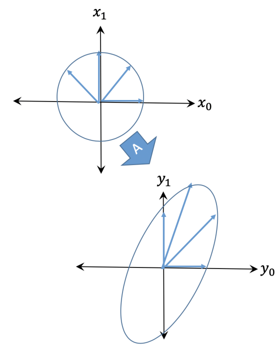
A linear transform $\vec{y} = A\vec{x}$ maps vector space $\vec{x}$ onto vector space $\vec{y}$. For example: the matrix $A = \left[\begin{array}{cc} 1 & 1 \\ 0 & 2 \end{array}\right]$ maps the vectors $\vec{x}_0, \vec{x}_1, \vec{x}_2, \vec{x}_3 =$

$$\left[\begin{array}{c} 1 \\ 0 \end{array}\right], \left[\begin{array}{c} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{array}\right], \left[\begin{array}{c} 0 \\ 1 \end{array}\right], \left[\begin{array}{c} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{array}\right]$$

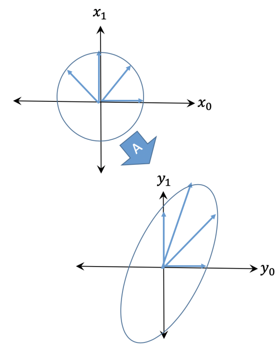
to the vectors $\vec{y}_0, \vec{y}_1, \vec{y}_2, \vec{y}_3 =$

$$\left[\begin{array}{c} 1 \\ 0 \end{array}\right], \left[\begin{array}{c} \sqrt{2} \\ \sqrt{2} \end{array}\right], \left[\begin{array}{c} 1 \\ 2 \end{array}\right], \left[\begin{array}{c} 0 \\ \sqrt{2} \end{array}\right]$$



A linear transform $\vec{y} = A\vec{x}$ maps vector space $\vec{x}$ onto vector space $\vec{y}$. The absolute value of the determinant of A tells you how much the area of a unit circle is changed under the transformation.

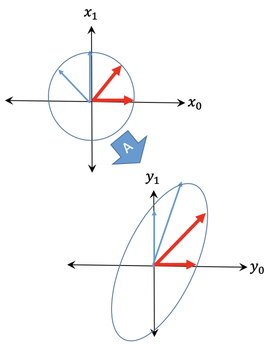
For example, if $A = \left[\begin{array}{cc} 1 & 1 \\ 0 & 2 \end{array}\right]$, then the unit circle in $\vec{x}$ (which has an area of π) is mapped to an ellipse with an area that is $\text{abs}(|A|) = 2$ times larger, i.e., i.e., $\pi\text{abs}(|A|) = 2\pi$.



For a D-dimensional square matrix, there may be up to D different directions $\vec{x} = \vec{v}_d$ such that, for some scalar λ_d, $A\vec{v}_d = \lambda_d \vec{v}_d$. For example, if $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$, then the eigenvectors are

$$\vec{v}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \vec{v}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix},$$

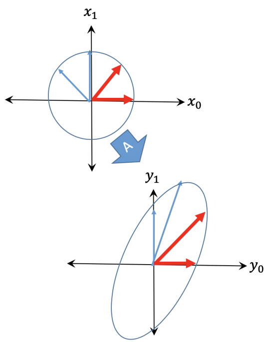
and the eigenvalues are $\lambda_0 = 1$, $\lambda_1 = 2$. Those vectors are red and extra-thick, in the figure to the left. Notice that one of the vectors gets scaled by $\lambda_0 = 1$, but the other gets scaled by $\lambda_1 = 2$.



An eigenvector is a direction, not just a vector. That means that if you multiply an eigenvector by any scalar, you get the same eigenvector: if $A\vec{v}_d = \lambda_d \vec{v}_d$, then its also true that $cA\vec{v}_d = c\lambda_d \vec{v}_d$ for any scalar c. For example: the following are the same eigenvector as $\vec{v}_1$

$$\sqrt{2}\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad -\vec{v}_1 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

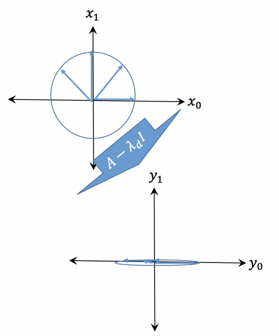
Since scale and sign don't matter, by convention, we normalize so that an eigenvector is always unit-length ($\|\vec{v}_d\| = 1$) and the first nonzero element is non-negative ($v_{d0} > 0$).



Eigenvalues: Before you find the eigenvectors, you should first find the eigenvalues. You can do that using this fact:

$$A\vec{v}_d = \lambda_d \vec{v}_d$$
$$A\vec{v}_d = \lambda_d I \vec{v}_d$$
$$A\vec{v}_d - \lambda_d I \vec{v}_d = \vec{0}$$
$$(A - \lambda_d I)\vec{v}_d = \vec{0}$$

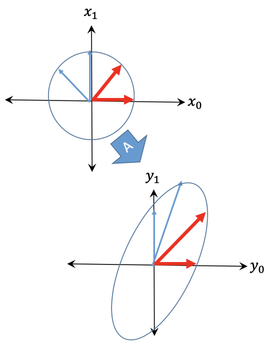
That means that when you use the linear transform $(A - \lambda_d I)$ to transform the unit circle, the result has an area of $|A - \lambda I| = 0$.



Example:

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix}$$
$$= 2 - 3\lambda + \lambda^2$$

which has roots at $\lambda_0 = 1$, $\lambda_1 = 2$



There are always D eigenvalues

- The determinant $|A - \lambda I|$ is a D^{th}-order polynomial in λ.
- By the fundamental theorem of algebra, the equation

$$|A - \lambda I| = 0$$

  has exactly D roots (counting repeated roots and complex roots).
- Therefore, **any square matrix has exactly D eigenvalues** (counting repeated eigenvalues, and complex eigenvalues.
- The same is not true of eigenvalues. **Not every square matrix has eigenvectors**. Complex and repeated eigenvalues usually correspond to eigensubspaces, not eigenvectors.

Outline

Summary

- DTFT of a rectangle:

$$x[n] = u[n] - u[n-N] \leftrightarrow X(\omega) = e^{-j\omega\left(\frac{N-1}{2}\right)} \frac{\sin(\omega N/2)}{\sin(\omega/2)}$$

- Jointly Gaussian RVs:

$$f_{\vec{X}}(\vec{x}) = \frac{1}{2\pi|\Sigma|^{1/2}} e^{-\frac{1}{2}(\vec{x}-\vec{\mu})^T\Sigma^{-1}(\vec{x}-\vec{\mu})}$$

- Linear algebra:

$$|A - \lambda I| = 0, \quad A\vec{v} = \lambda\vec{v}$$