UNIVERSITY OF ILLINOIS
Department of Electrical and Computer Engineering
ECE 417 Multimedia Signal Processing

Lecture 12 Sample Problem Solutions

Problem 12.1

1. If we apply the inverse 2D DTFT formula,

$$h[n_1,n_2]=\frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}H(\omega_1,\omega_2)e^{j\omega_1 n_2}e^{j\omega_2 n_2}d\omega_1 d\omega_2$$

We get that

$$h[n_1,n_2]=\left(\frac{1}{5}\text{sinc}\left(\frac{\pi n_1}{5}\right)\right)\left(\frac{1}{3}\text{sinc}\left(\frac{\pi n_2}{3}\right)\right)$$

for all $-\infty < n_1 < \infty$ and $-\infty < n_2 < \infty$.

2. The standard way to implement 2D convolution is

$$f[n_1,n_2]=\sum_{m_1=\infty}^{\infty}\sum_{m_2=-\infty}^{\infty}x[m_1,m_2]h[n_1-m_1,n_2-m_2] \qquad (12.1\text{-}1)$$

Eq. 12.1-1 requires a double-summation, with N_1N_2 terms, in order to compute each output pixel. There are $(N_1+M_1-1)\times(N_2+M_2-1)=\mathcal{O}\left\{N_1N_2\right\}$ output pixels. So the total complexity is $\mathcal{O}\left\{(N_1N_2)^2\right\}$.

More loosely, if $N_1\approx N_2\approx N$, then we can say that standard convolution has a complexity of $\mathcal{O}\left\{N^4\right\}$.

3. The filter $h[n_1,n_2]$ is separable, so we can filter the rows first, and then the columns:

$$f[n_1,n_2]=\sum_{m_1=\infty}^{\infty}\left(\sum_{m_2=-\infty}^{\infty}x[m_1,m_2]h_2[n_2-m_2]\right)h_1[n_1-m_1] \qquad (12.1\text{-}2)$$

where

$$h_1[n_1]=\left(\frac{1}{5}\text{sinc}\left(\frac{\pi n_1}{5}\right)\right),\quad h_2[n_2]=\left(\frac{1}{3}\text{sinc}\left(\frac{\pi n_2}{3}\right)\right)$$

Eq. 12.1-2 does the row-convolution first, requiring N_2 operations per pixel. Then it does the column-convolution, requiring N_1 operations per pixel. If there are $\mathcal{O}\left\{N_1N_2\right\}$ pixels, then the complexity is $\mathcal{O}\left\{N_1N_2(N_1+N_2)\right\}$.

Problem 12.2

Let's assume that the spectrum of $u[n]$ is

$$U(\omega)=\frac{1}{|\omega|}$$

for $|\omega|<\pi$. After filtering with an ideal anti-aliasing filter, it is

$$V(\omega)=\begin{cases}\frac{1}{|\omega|} & |\omega|\le\frac{\pi}{2}\\ 0 & \text{otherwise}\end{cases}$$

After downsampling, we get

$$X(\omega) = \frac{1}{2}\sum_{k=0}^{1} V(\frac{\omega - 2\pi k}{2}) = \frac{1}{2|\omega/2|} = \frac{1}{|\omega|}$$

After upsampling,

$$Y(\omega) = X(2\omega) = \begin{cases} \frac{1}{2|\omega|} & |\omega| \le \frac{\pi}{2} \\ \frac{1}{2(\pi-|\omega|)} & \frac{\pi}{2} \le |\omega| \le \pi \end{cases}$$

1. If $h_a[n] = \frac{\sin(\pi n/2)}{\pi n/2}$, then

   $$H_a(\omega) = \begin{cases} 2 & |\omega| \le \frac{\pi}{2} \\ 0 & \text{otherwise} \end{cases}$$

   So

   $$Z(\omega) = H_a(\omega)Y(\omega) = \begin{cases} \frac{1}{|\omega|} & |\omega| \le \frac{\pi}{2} \\ 0 & \frac{\pi}{2} \le |\omega| \le \pi \end{cases}$$

   Which is exactly equal to $U(\omega)$ for $|\omega| \le \pi/2$, and exactly equal to zero at higher frequencies.

2. If $h_b[n] = (h_a[n])^2$, then

   $$H_b(\omega) = \frac{1}{2\pi} H_a(\omega) \circledast H_a(\omega) = 2\left(\frac{\pi - |\omega|}{\pi}\right)$$

   and therefore

   $$Z(\omega) = H_b(\omega)Y(\omega) = \begin{cases} \frac{(\pi-|\omega|)/\pi}{|\omega|} & |\omega| \le \frac{\pi}{2} \\ \frac{1}{\pi} & \frac{\pi}{2} \le |\omega| \le \pi \end{cases}$$

   which is slightly less than $U(\omega)$ at low frequencies ($(\pi - |\omega|)/pi$ times less), but then slightly higher than $U(\omega)$ at high frequencies (π/ω times higher). Actually, the spectral shape in general is closer to the spectral shape of $U(\omega)$, so that, even though the high frequencies are entirely constructed from aliasing, it is still often true that this version of $Z(\omega)$ is a better-looking image than the ideal band-limited version.

   An even better interpolating filter can be constructed using a two-part interpolator, $H_c(\omega)$, which has one type of spectrum below $\pi/2$, and a different nonzero type of spectrum at high frequencies.

   An even better solution is to estimate the high frequencies using a nonlinear regression algorithm, e.g., a neural net trained to estimate the high-frequency component of $U(\omega)$ given its low-frequency component.