UNIVERSITY OF ILLINOIS
Department of Electrical and Computer Engineering
ECE 417 Multimedia Signal Processing

Lecture 5 Sample Problems

Problem 5.1

Suppose that a particular covariance matrix, $R = V\Lambda V^T$, has a trace equal to $D\sigma^2$, and eigenvalues given by

$$\lambda_d = D\sigma^2 \left(\frac{1}{2}\right)^n, \quad 1 \le d \le D-1$$

and $\lambda_D = \lambda_{D-1}$. Suppose that $D = 512$ dimensions.

The principal components are defined as

$$\vec{y} = [\vec{v}_1, \ldots, \vec{v}_M]^T (\vec{x} - \vec{\mu})$$

where $M < D$ is the number of principal components retained. What is the smallest value of M that will result in $E\left[\|\vec{y}\|^2\right] \ge 0.95 D\sigma^2$?

Problem 5.2

What is the probability density function of the vector $\vec{y}$ in problem 1?

Problem 5.3

Consider a particular covariance matrix, $R = V\Lambda V^T$, where $V = [\vec{v}_1, \ldots, \vec{v}_D]$, and the vectors $\vec{v}_d$ are orthonormal. Suppose that the first C eigenvalues, $\lambda_1, \ldots, \lambda_C$ are all positive, but the remaining $D - C$ eigenvalues are all zero.

A matrix whose eigenvalues are all $\lambda_d \ge 0$ is called a "positive semi-definite matrix." If some eigenvalues are zero, there is no matrix R^{-1} such that $R^{-1}R = I$. It's possible, however, to define a pseudo-inverse $R^\dagger$ that has some of the properties of an inverse, for example:

1. $R^\dagger R = \sum_{m=1}^{C} \vec{v}_m \vec{v}_m^T$
2. $R^\dagger R R^\dagger = R^\dagger$, and
3. $R R^\dagger R = R$

Come up with a definition of $R^\dagger$, in terms of the nonzero eigenvalues λ_d and their corresponding eigenvectors $\vec{v}_d$, that satisfies these three requirements.

Problem 5.4

Suppose you have a two-class classification problem, with D-dimensional observations given by

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_D \end{bmatrix}$$

In this case, use the random variable $Y \in \{0, 1\}$ as the class label (e.g., the name of the person you're trying to identify). The prior probabilities are given by the known parameter π_0:

$$p_Y(0) = \pi_0, \quad p_Y(1) = 1 - \pi_0$$

Suppose that, if you know the class label, the feature vector has no further randomness at all. If vector $\vec{x}$ is drawn from class $Y = 0$, then it ALWAYS has a value of $\vec{x} = \vec{\mu}_0$; if it is drawn from class $Y = 1$, then it ALWAYS has a value of $\vec{x}_n = \vec{\mu}_1$.

Define the global mean, covariance, and principal components to be

$$\vec{\mu} = E\left[\vec{x}\right], \quad R = E\left[\left(\vec{x} - \vec{\mu}\right)\left(\vec{x} - \vec{\mu}\right)^T\right], \quad R = V\Lambda V^T, \quad V^T V = I, \quad \Lambda \text{ diagonal}$$

Find $\vec{\mu}$, R, V and Λ in terms of the parameters π_0, $\vec{\mu}_0$, and $\vec{\mu}_1$.