• This is a CLOSED BOOK exam. You may use one sheet (front and back) of handwritten notes.

• No calculators are permitted. You need not simplify explicit numerical expressions.

• There are a total of 50 points in the exam. Each problem specifies its point total. Plan your work accordingly.

• You must SHOW YOUR WORK to get full credit.

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Possibly Useful Formulas

YPbPr and Sobel Mask

\[
\begin{bmatrix}
Y \\
P_b \\
P_r
\end{bmatrix} =
\begin{bmatrix}
0.299 & 0.587 & 0.114 \\
-0.168736 & -0.331264 & 0.5 \\
0.5 & -0.418688 & -0.081312
\end{bmatrix}
\begin{bmatrix}
R \\
G \\
B
\end{bmatrix}
\]

\[G_x[n_1, n_2] = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \\ 1 & 0 & -1 \end{bmatrix} \ast I[n_1, n_2], \quad G_y[n_1, n_2] = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ -1 & -2 & -1 \end{bmatrix} \ast I[n_1, n_2]\]

Integral Image and Lowpass Filter

\[G[n_1, n_2] = \sum_{m_1=0}^{n_1} \sum_{m_2=0}^{n_2} i[m_1, m_2] \]

\[H(\omega_1, \omega_2) = \begin{cases} 1 & |\omega_1| < \phi_1, \ |\omega_2| < \phi_2 \\ 0 & \text{otherwise} \end{cases}, \quad h[n_1, n_2] = \left( \frac{\phi_1}{\pi} \right) \left( \frac{\phi_2}{\pi} \right) \text{sinc} (\phi_1 n_1) \text{sinc} (\phi_2 n_2)\]

Orthogonality Principle and LPC

\[\varepsilon = E \left[ \left( x[n] - \sum_{m=1}^{p} \alpha_m x[n-m] \right)^2 \right], \quad \frac{\partial \varepsilon}{\partial \alpha_k} = -2E \left[ (x[n-k] \left( x[n] - \sum_{m=1}^{12} \alpha_m x[n-m] \right) \right] \]

\[R_{xx}[k] = \sum_{m=1}^{12} \alpha_m R_{xx}[k-m] \]

Fourier Series

\[x[n] = \sum_{k=0}^{P-1} X_k e^{j2\pi kn/P} \quad X_k = \frac{1}{P} \sum_{n=0}^{P-1} x[n] e^{-j2\pi kn/P} \]

Autocorrelation and Power Spectrum

\[R_{xx}[n] = E \{ x[m] x[m-n] \} \leftrightarrow S_{xx}(\omega) = \sum_{n=-\infty}^{\infty} R_{xx}[n] e^{-j\omega n} \]

\[r_{xx}[n] = \sum_{m=-\infty}^{\infty} x[m] x[m-n] \leftrightarrow s_{xx}(\omega) = \sum_{n=-\infty}^{\infty} r_{xx}[n] e^{-j\omega n} \]
Problem 1  (10 points)

Consider an unvoiced zero-mean Gaussian random signal, $x[n] \sim \mathcal{N}(0, 1)$, with the following autocorrelation:

$$R_{xx}[n] = e^{-\beta |n|}$$

Suppose $e[n] = x[n] - \alpha x[n - 1]$.

(a) Find $\alpha$ to minimize $E[e^2[n]]$.

(b) Find the value of $E[e^2[n]]$ that results from the $\alpha$ you chose in part (a).
Problem 2  (5 points)

Consider the synthesis filter \( s[n] = e[n] + bs[n - 1] - \left( \frac{b}{2} \right)^2 s[n - 2] \). For what values of \( b \) is the synthesis filter stable?
Problem 3  (5 points)

Suppose $e[n] \sim \mathcal{N}(0, 1)$ is Gaussian white noise, $R_{ee}[n] = \delta[n]$. Consider the synthesis filter, $s[n] = e[n] + \alpha s[n-1]$. Find the power spectrum of the synthesized signal, $S_{ss}(\omega)$, in terms of $\omega$ and $\alpha$. You need not simplify, but your answer should contain no integrals or infinite sums.
Problem 4  (5 points)

You are given the integral image $ii[n_1,n_2]$, defined in terms of the image $i[n_1,n_2]$ as

$$ii[n_1,n_2] = \sum_{m_1=0}^{n_1} \sum_{m_2=0}^{n_2} i[m_1,m_2]$$

Write an equation that computes the complementary integral image, $c[n_1,n_2]$, in a small constant number of operations per output pixel, where

$$c[n_1,n_2] = \sum_{m_1=n_1}^{N_1-1} \sum_{m_2=n_2}^{N_2-1} i[m_1,m_2]$$
Problem 5  (5 points)

Suppose you have a $200 \times 200$-pixel image that is just one white dot at pixel $(45, 25)$, and all the other pixels are black:

$$x[n_1, n_2] = \begin{cases} 
255 & n_1 = 45, \ n_2 = 25 \\
0 & \text{otherwise, } 0 \leq n_1 < 199, \ 0 \leq n_2 < 199 
\end{cases}$$

This image is upsampled to size $400 \times 400$, then filtered, as

$$y[n_1, n_2] = \begin{cases} 
x[n_1/2, n_2/2] & n_1/2 \text{ and } n_2/2 \text{ both integers} \\
0 & \text{otherwise}
\end{cases} \quad z[n_1, n_2] = y[n_1, n_2] \ast h[n_1, n_2]$$

where $h[n_1, n_2]$ is the ideal anti-aliasing filter whose frequency response is

$$H(\omega_1, \omega_2) = \begin{cases} 
1 & |\omega_1| < \frac{\pi}{2}, \ |\omega_2| < \frac{\pi}{2} \\
0 & \text{otherwise}
\end{cases}$$

Find $z[n_1, n_2]$. 
Problem 6  (10 points)

Consider an infinite-sized RGB image containing a single diagonal white line on a black background, specifically

\[ R[n_1, n_2] = G[n_1, n_2] = B[n_1, n_2] = \begin{cases} 255 & n_1 - n_2 = 5 \\ 0 & \text{otherwise} \end{cases} \]

where \(-\infty < n_1 < \infty, -\infty < n_2 < \infty\), and the signals \(R, G,\) and \(B\) are the red, green, and blue channels, respectively.

(a) Find the luminance \(Y[n_1, n_2]\).

(b) Find the blue-shift \(P_b[n_1, n_2]\).
Problem 7  (10 points)

Consider an infinite-sized grayscale image of a diagonal gray line:

\[
x[n_1, n_2] = \begin{cases} 
105 & n_1 - n_2 = 5 \\
0 & \text{otherwise}
\end{cases}, \quad -\infty < n_1 < \infty, \quad -\infty < n_2 < \infty
\]

**NOTE:** EACH part of this problem is a **1D ROW CONVOLUTION** of the ORIGINAL IMAGE with a **DIFFERENT** row filter. Although these filters are related to the Sobel mask, NEITHER PART OF THIS PROBLEM IMPLEMENTS A COMPLETE SOBEL MASK.

(a) Suppose we **convolve each row** with a differencing filter:

\[
y[n_1, n_2] = x[n_1, n_2] * d_2[n_2], \quad d_2[n_2] = \begin{cases} 
1 & n_2 = 0 \\
-1 & n_2 = 2 \\
0 & \text{otherwise}
\end{cases}
\]

Find \(y[n_1, n_2]\).

(b) Suppose, INSTEAD, that we **convolve each row** with an averaging filter

\[
z[n_1, n_2] = x[n_1, n_2] * a_2[n_2], \quad a_2[n_2] = \begin{cases} 
1 & n_2 \in \{0, 2\} \\
2 & n_2 = 1 \\
0 & \text{otherwise}
\end{cases}
\]

Find \(z[n_1, n_2]\).