This is a CLOSED BOOK exam. You may use one sheet (front and back) of handwritten notes.

No calculators are permitted. You need not simplify explicit numerical expressions.

There are a total of 50 points in the exam. Each problem specifies its point total. Plan your work accordingly.

You must SHOW YOUR WORK to get full credit.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td></td>
</tr>
</tbody>
</table>

Name: ________________________________
Possibly Useful Formulas

Fourier Transforms

\[ X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \leftrightarrow x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n} d\omega \]

\[ X[k] = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi kn}{N}} \leftrightarrow x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[j]e^{j\frac{2\pi kn}{N}} \]

\[ x[n] = e^{j\omega n} \leftrightarrow X(e^{j\omega}) = 2\pi \delta(\omega - \omega_0) \]

\[ w_R[n] = \begin{cases} 1 & 0 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases} \leftrightarrow W_R(\omega) = \frac{\sin(\omega N/2)}{\sin(\omega/2)}e^{-j\omega(\frac{N+1}{2})} \]

Autocorrelation and Power Spectrum

\[ R_{xx}[n] = E\{x[m]x[m-n]\} \leftrightarrow S_{xx}(\omega) = \sum_{n=-\infty}^{\infty} R_{xx}[n]e^{-j\omega n} \]

\[ r_{xx}[n] = \sum_{m=-\infty}^{\infty} x[m]x[m-n] \leftrightarrow s_{xx}(\omega) = \sum_{n=-\infty}^{\infty} r_{xx}[n]e^{-j\omega n} \]

Gaussians, Mahalanobis, and PCA

\[ \mathcal{N} (\vec{x}; \vec{\mu}, \Sigma) = \frac{1}{(2\pi)^{D/2}|R|^{1/2}}e^{-\frac{1}{2}d_R^2(\vec{x},\vec{\mu})} \]

\[ R = V\Lambda V^T, \quad V^TV =VV^T = I, \quad |R| = |\Lambda| \]

\[ d_R^2(\vec{x},\vec{\mu}) = (\vec{x} - \vec{\mu})^T R^{-1}(\vec{x} - \vec{\mu}) = \vec{y}^T \Lambda^{-1} \vec{y}, \quad \vec{y} = V^T(\vec{x} - \vec{\mu}) \]
Problem 1  (20 points)

A particular signal, $x[n]$, is sampled at $F_s = 18,000$ samples/second. There are a total of 10,000 samples, numbered $x[0]$ through $x[9999]$. These samples are divided into $T$ frames, $\vec{x}_t$, with a framelength of 250 samples and a frame skip of 100 samples, i.e.,

$$\vec{x}_t = \begin{bmatrix} x[100t] \\ \vdots \\ x[100t + 249] \end{bmatrix}$$

Your goal is to create two different $480 \times T$ matrices: $X = [\vec{X}_0, \ldots, \vec{X}_{T-1}]$ is the STFT (short-time Fourier transform) of $x[n]$, and $S = [\vec{S}_0, \ldots, \vec{S}_{T-1}]$ is the spectrogram of $x[n]$. The final image matrix $S$ should show the spectral level (in decibels) of $x[n]$, as a function of time and frequency, in the frequency range from 0Hz to 5000Hz.

(a) Find $T$, the number of frames. This should be set so that (1) every sample of $x[n]$ appears in at least one frame, and (2) there is at most one frame with zero-padding. Your answer should be a number, or an explicit numerical expression.

(b) Each STFT vector, $\vec{X}_t$, is the length-$N$ DFT of one frame $\vec{x}_t$. Find $N$. Your answer should be a number, or an explicit numerical expression.
(c) The STFT is given by $\tilde{X}_t = A\tilde{x}_t$ for some matrix, $A$, whose $(k,n)^{th}$ element is $a_{kn}$. Give an expression for $a_{kn}$ in terms of $k$, $n$, and $N$.

(d) Suppose that $X_{max} = \max_k \max_t |X[k,t]|$. The spectrogram $S[k,t]$ is the level of $X[k,t]$, in decibels, scaled so that $0 \leq S[k,t] \leq 255$, and so that $S[k,t] = 0$ if and only if $|X[k,t]| \leq X_{max}/1000$. Give an equation specifying $S[k,t]$ as a function of $X[k,t]$.
Problem 2  (5 points)

The signal \( x[n] \) is given by

\[
x[n] = \begin{cases} 
\cos(\omega_0 n) & 0 \leq n \leq N - 1 \\
0 & \text{otherwise}
\end{cases}
\]

\( X[k] \) is the length-\( N \) DFT of \( x[n] \). Find \( X[k] \), in terms of \( N \) and \( \omega_0 \). You may find it useful to write your answer in terms of the transform of a rectangular window, \( W_R(\omega) \), which is

\[
w_R[n] = \begin{cases} 
1 & 0 \leq n \leq N - 1 \\
0 & \text{otherwise}
\end{cases} \quad \leftrightarrow \quad W_R(\omega) = \frac{\sin(\omega N/2)}{\sin(\omega/2)} e^{-j\omega (\frac{N-1}{2})}
\]
Problem 3  (5 points)

A random signal $x[n]$ has the autocorrelation function $R_{xx}[n] = \rho^{|n|}$, for some real constant $0 < \rho < 1$. Find its power spectrum $S_{xx}(\omega)$, in terms of $\omega$ and $\rho$. Your answer should contain no infinite-length summations.
Problem 4  (5 points)

$x[n]$ is a signal with $N$ samples, numbered $x[0]$ through $x[N - 1]$. Find $M$ and $s[m]$ so that

(a) every sample of $s[m]$ is either $s[m] = 0$ or $s[m] = x[n]$ for some $n$,

(b) every sample of $x[n]$ is used at least once,

(c) $S[k]$, the $M$-point DFT of $s[n]$, is real-valued.

You may define the sample times $m$ to be non-integers, if you wish, though correct answers with integer-valued sample times also exist.
Problem 5 (15 points)

Suppose you have an \(M \times D\) matrix, \(X = [\vec{x}_0, \ldots, \vec{x}_{M-1}]^T\), where \(\sum_{m=0}^{M-1} \vec{x}_m = \vec{0}\). The eigenvalues of \(X^TX\) are \(\lambda_0\) through \(\lambda_{D-1}\), its eigenvectors are \(\vec{v}_0\) through \(\vec{v}_{D-1}\), and its principal components are \(Y = XV\).

(a) Write \(Y^TY\) in terms of the eigenvalues, \(\lambda_0\) through \(\lambda_{D-1}\).

(b) Write \(\sum_{m=0}^{M-1} \|\vec{x}_m\|^2\) in terms of the eigenvalues, \(\lambda_0\) through \(\lambda_{D-1}\).
(c) Write $\vec{v}_i^T X^T X \vec{v}_j$ in terms of the eigenvalues, $\lambda_0$ through $\lambda_{D-1}$, for $0 \leq i \leq j \leq D - 1$. 