UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN
Department of Electrical and Computer Engineering

ECE 417 Multimedia Signal Processing
Fall 2018

EXAM 1 SOLUTIONS

Thursday, October 18, 2018

Problem 1 (16 points)

A particular random signal $u[n]$ has the following DTFT:

$$U(e^{j\omega}) = ae^{j\theta}\delta(\omega - 0.2\pi) + ae^{-j\theta}\delta(\omega + 02.\pi)$$

where

- a is a real-valued Gaussian random variable with mean 0 and variance σ^2
- θ is a real-valued random variable uniformly distributed between 0 and 2π.

Find the random signal $u[n]$, and its statistical autocorrelation $R_{uu}[m]$, in terms of a, σ^2, θ, n, and/or m.

Solution

$$\begin{aligned} u[n] &= \frac{a}{2\pi}e^{j\theta}e^{j0.2\pi n} + \frac{a}{2\pi}e^{-j\theta}e^{-j0.2\pi n} \\ &= \frac{a}{\pi}\cos\left(\theta + 0.2\pi n\right) \end{aligned}$$

$$\begin{aligned} R_{uu}[m] &= E\left[u[n]u[n-m]\right] \\ &= E\left[\left(\frac{a}{\pi}\right)^2 \cos\left(\theta + 0.2\pi n\right)\cos\left(\theta + 0.2\pi(n-m)\right)\right] \\ &= E\left[\frac{a^2}{2\pi^2}\cos(2\theta + 0.2\pi(2n-m))\right] + E\left[\frac{a^2}{2\pi^2}\cos(0.2\pi m)\right] \\ &= 0 + \frac{\sigma^2}{2\pi^2}\cos(0.2\pi m) \end{aligned}$$

Problem 2 (17 points)

A particular voiced speech signal has pitch period P, and vocal tract transfer function $H(e^{j\omega})$. The signal is windowed by a window function $w[n]$ of length N, producing the windowed signal

$$s[n] = \begin{cases} w[n]\sum_{\ell=-\infty}^{\infty} h[n-\ell P] & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$$

Find $S[k]$, the N-point DFT of $s[n]$, in terms of k, P, N, $H(e^{j\omega})$, and $W(e^{j\omega})$.

Solution

$s[n] = w[n]\left(e[n] * h[n]\right)$, where

$$e[n] = \sum_{\ell=-\infty}^{\infty} \delta[n-\ell P] \leftrightarrow E(e^{j\omega}) = \left(\frac{2\pi}{P}\right)\sum_{m=0}^{P-1}\delta\left(\omega - \frac{2\pi m}{P}\right)$$

$$\begin{aligned} S(e^{j\omega}) &= \frac{1}{2\pi}W(e^{j\omega}) \circledast \left(H(e^{j\omega})E(e^{j\omega})\right) \\ &= \frac{1}{2\pi}W(e^{j\omega}) \circledast \left(\frac{2\pi}{P}\sum_{m=0}^{P-1} H(e^{j\frac{2\pi m}{P}})\delta\left(\omega - \frac{2\pi m}{P}\right)\right) \\ &= \frac{1}{P}\sum_{m=0}^{P-1} H(e^{j\frac{2\pi m}{P}})W\left(e^{j\left(\omega-\frac{2\pi m}{P}\right)}\right) \end{aligned}$$

$$S[k] = \frac{1}{P}\sum_{m=0}^{P-1} H(e^{j\frac{2\pi m}{P}})W\left(e^{j\left(\frac{2\pi k}{N}-\frac{2\pi m}{P}\right)}\right)$$

Problem 3 (17 points)

Your goal is to find a positive real number, a, so that $ax[n]$ is as similar as possible to $y[n]$ in the sense that it minimizes the following error:

$$\epsilon = \int_{-\pi}^{\pi}\left(|Y(e^{j\omega})| - a|X(e^{j\omega})|\right)^2 d\omega$$

Find the value of a that mininimizes ϵ, in terms of $|X(e^{j\omega})|$ and $|Y(e^{j\omega})|$.

Solution:

$$\frac{\partial\epsilon}{\partial a} = 2\int_{-\pi}^{\pi}\left(a|X(e^{j\omega})| - |Y(e^{j\omega})|\right)|X(e^{j\omega})|d\omega$$

which equals 0 at:

$$a = \frac{\int_{-\pi}^{\pi}|X(e^{j\omega})||Y(e^{j\omega})|d\omega}{\int_{-\pi}^{\pi}|X(e^{j\omega})|^2 d\omega}$$

Problem 4 (17 points)

A 2-dimensional Gaussian random vector has mean $\vec{\mu}$ and covariance Σ given by

$$\vec{\mu} = \left[\begin{array}{c}1\\1\end{array}\right],\quad \Sigma = \left[\begin{array}{cc}\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\end{array}\right]\left[\begin{array}{cc}8 & 0\\ 0 & 2\end{array}\right]\left[\begin{array}{cc}\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\end{array}\right]$$

Draw a curve of some kind, on a two-dimensional Cartesian plane, showing the set of points $\left\{\vec{x} : p_X(\vec{x}) = \frac{1}{8\pi}e^{-\frac{1}{2}}\right\}$.

Solution:

$|\Sigma| = |\Lambda| = 16$, so

$$p_X(\vec{x}) = \frac{1}{8\pi} e^{-\frac{1}{2}d^2_\Sigma(\vec{x},\vec{\mu})}$$

so the solution is the set $\left\{\vec{x} : d^2_\Sigma(\vec{x},\vec{\mu}) = 1\right\}$.

$$d^2_\Sigma(\vec{x},\vec{\mu}) = \vec{y}^T\Lambda^{-1}\vec{y} = \frac{y_1^2}{8} + \frac{y_2^2}{2}$$

$$\vec{y} = \frac{\sqrt{2}}{2}\left[\begin{array}{c}(x_1-1)+(x_2-1)\\(x_1-1)-(x_2-1)\end{array}\right]$$

So the solution is the set

$$\left\{\vec{x} : \frac{(x_1+x_2-2)^2}{16} + \frac{(x_1-x_2)^2}{4} = 1\right\}$$

...which is an ellipse, centered at $(1,1)$, with a radius of $2\sqrt{2}$ along the $(1,1)$ direction, and a radius of $\sqrt{2}$ along the $(1,-1)$ direction.

Problem 5 (16 points)

In terms of $\alpha_t(i)$, $\beta_t(i)$, a_{ij}, π_i and $b_i(\vec{x}_t)$, find

$$p\left(q_6 = i, q_7 = j|\vec{x}_1,\ldots,\vec{x}_{20}\right)$$

Solution:

$$\begin{aligned} p(q_6 = i, q_7 = j, \vec{x}_1,\ldots,\vec{x}_{20}) &= p(\vec{x}_1,\ldots,\vec{x}_6, q_6 = i)p(q_7 = j|q_6 = i)p(\vec{x}_7|q_7 = j)p(\vec{x}_8,\ldots,\vec{x}_{20}|q_7 = j) \\ &= \alpha_6(i)a_{ij}b_j(\vec{x}_7)\beta_7(j) \end{aligned}$$

$$p\left(q_6 = i, q_7 = j|\vec{x}_1,\ldots,\vec{x}_{20}\right) = \frac{\alpha_6(i)a_{ij}b_j(\vec{x}_7)\beta_7(j)}{\sum_{k=1}^N\sum_{\ell=1}^N \alpha_6(k)a_{k\ell}b_\ell(\vec{x}_7)\beta_7(\ell)}$$

Problem 6 (17 points)

A particular HMM-based speech recognizer only knows two words: word w_0, and word w_1. Word w_0 has a higher *a priori* probability: $p_Y(w_0) = 0.7$, while $p_Y(w_1) = 0.3$. Each of the two words is modeled by a four-state Gaussian HMM ($N = 4$) with three-dimensional observations ($D = 3$). All states, in both HMMs, have identity covariance ($\Sigma_i = I$). Both HMMs have *exactly* the same transition probabilities and state-dependent means, given by:

Both Words: $A = \left[\begin{array}{cccc}0.25 & 0.25 & 0.25 & 0.25\\0.25 & 0.25 & 0.25 & 0.25\\0.25 & 0.25 & 0.25 & 0.25\\0.25 & 0.25 & 0.25 & 0.25\end{array}\right]$, $\vec{\mu}_1 = \left[\begin{array}{c}0\\0\\0\end{array}\right]$, $\vec{\mu}_2 = \left[\begin{array}{c}1\\1\\1\end{array}\right]$, $\vec{\mu}_3 = \left[\begin{array}{c}1\\-1\\-1\end{array}\right]$, $\vec{\mu}_4 = \left[\begin{array}{c}-1\\-1\\1\end{array}\right]$

But the initial residence probabilities are different:

$$\textbf{Word 0:}\ \pi_i = \begin{cases} 1 & i=1 \\ 0 & \text{otherwise} \end{cases} \qquad \textbf{Word 1:}\ \pi_i = \begin{cases} 1 & i=4 \\ 0 & \text{otherwise} \end{cases}$$

Suppose that you have a two-frame observation, $X = [\vec{x}_1, \vec{x}_2]$, where $\vec{x}_t = [x_{1t}, x_{2t}, x_{3t}^T]$. The MAP decision rule, in this case, can be written as a linear classifier,

$$\hat{y} = \begin{cases} w_1 & \vec{w}_1^T\vec{x}_1 + \vec{w}_2^T\vec{x}_2 + b > 0 \\ w_0 & \text{otherwise} \end{cases}$$

Find $\vec{w}_1$, $\vec{w}_2$, and b.

Solution:

The Bayesian classifier chooses w_1 if

$$\begin{aligned} p(w_0)p(X|w_0) &< p(w_1)p(X|w_1) \\ 0.7\mathcal{N}(\vec{x}_1|\vec{\mu}_1)\sum_j a_{1j}\mathcal{N}(\vec{x}_2|\vec{\mu}_j) &< 0.3\mathcal{N}(\vec{x}_4|\vec{\mu}_4)\sum_j a_{4j}\mathcal{N}(\vec{x}_2|\vec{\mu}_j) \\ 0.7\mathcal{N}(\vec{x}_1|\vec{\mu}_1) &< 0.3\mathcal{N}(\vec{x}_1|\vec{\mu}_4) \\ \ln(0.7) - \frac{1}{2}(\vec{x}_1-\vec{\mu}_1)^T(\vec{x}_1-\vec{\mu}_1) &< \ln(0.3) - \frac{1}{2}(\vec{x}_1-\vec{\mu}_4)^T(\vec{x}_1-\vec{\mu}_4) \\ \ln(0.7) - \frac{1}{2}\|\vec{x}_1\|^2 &< \ln(0.3) - \frac{1}{2}\|\vec{x}_1\|^2 + \vec{\mu}_4^T\vec{x}_1 - \frac{1}{2}\|\vec{\mu}_4\|^2 \end{aligned}$$

Which is satisfied if

$$\vec{\mu}_4^T\vec{x}_1 + \ln\left(\frac{3}{7}\right) - \frac{3}{2} > 0$$

So

$$\vec{w}_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \quad \vec{w}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad b = \ln\left(\frac{3}{7}\right) - \frac{3}{2}$$