UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN
Department of Electrical and Computer Engineering

ECE 417 Multimedia Signal Processing
Fall 2017

EXAM 1

Thursday, September 28, 2017

- This is a CLOSED BOOK exam. You may use one sheet (front and back) of hand-written notes.
- No calculators are permitted. You need not simplify explicit numerical expressions.
- There are a total of 40 points in the exam. Each problem specifies its point total. Plan your work accordingly.
- You must SHOW YOUR WORK to get full credit.

Problem	Score
1	
2	
3	
4	
Total	

Name: ____

Possibly Useful Formulas

Minkowski Norm

$$\|\vec{x}-\vec{\mu}\|_p = \left(|x_1-\mu_1|^p + \ldots + |x_D-\mu_D|^p\right)^{1/p}$$

Gaussians

$$\mathcal{N}(\vec{x};\vec{\mu},\Sigma) = \frac{1}{(2\pi)^{D/2}|\Sigma|^{1/2}} e^{-\frac{1}{2}(\vec{x}-\vec{\mu})^T\Sigma^{-1}(\vec{x}-\vec{\mu})}$$

$$\begin{aligned} \Sigma &= U\Lambda U^T \\ \Sigma^{-1} &= U\Lambda^{-1}U^T \\ U^T\Sigma U &= \Lambda \\ U^T U &= I \end{aligned}$$

Mahalanobis Distance and PCA

$$d_\Sigma^2(\vec{x},\vec{\mu}) = (\vec{x}-\vec{\mu})^T\Sigma^{-1}(\vec{x}-\vec{\mu}) = \vec{y}^T\Lambda^{-1}\vec{y}$$

$$\vec{y} = U^T(\vec{x}-\vec{\mu})$$

Bayesian Classifier

$$\hat{y} = \arg\max p_{Y|\vec{X}}(y|\vec{x})$$

Problem 1 (10 points)

Suppose you have a dataset including the vectors

$$\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \quad \vec{y} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}, \quad \vec{z} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

(a) Express $\|\vec{y} - \vec{z}\|_p$ as a function of p.

(b) Find a diagonal matrix Σ such that $d^2_\Sigma(\vec{x}, \vec{y}) > d^2_\Sigma(\vec{x}, \vec{z})$.

Problem 2 (10 points)

Define $\Phi(z)$ as follows:

$$\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du$$

Suppose $\vec{X} = [X_1, X_2]^T$ is a Gaussian random vector with mean and covariance given by

$$\vec{\mu} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}$$

(a) Sketch the set of points such that $f_{\vec{X}}(\vec{x}) = \frac{1}{12\pi} e^{-\frac{1}{8}}$, where $f_{\vec{X}}(\vec{x})$ is the pdf of $\vec{X}$.

(b) In terms of $\Phi(z)$, find the probability $\Pr\{-1 < X_1 < 1, -1 < X_2 < 1\}$.

Problem 3 (10 points)

Suppose that a particular covariance matrix Σ has the following eigenvector matrix, U, and eigenvalue matrix, Λ:

$$U = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$$

Let $\vec{y}(\vec{x}) = \begin{bmatrix} y_1(\vec{x}) \\ y_2(\vec{x}) \end{bmatrix} = U^T\vec{x}$ be the principal components of a vector space $\vec{x}$.

(a) Plot the set of vectors $\vec{x}$ such that $y_1(\vec{x}) = 3$.

(b) Find the squared Mahalanobis distance, $d^2_\Sigma(\vec{x}, \vec{\mu})$, between the vectors $\vec{x}$ and $\vec{\mu}$ where

$$\vec{x} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}, \quad \vec{\mu} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Problem 4 (10 points)

Suppose that, for a particular classification problem, the correct label of every data point is as follows:

$$y^*(\vec{x}) = \begin{cases} 1 & \|\vec{x}\|_2 < 1.5 \\ 0 & \|\vec{x}\|_2 > 1.5 \end{cases} \tag{1}$$

Unfortunately, you aren't allowed to use the correct labeling function. Instead, you have to try to learn a nearest-neighbor or Bayesian classifier.

(a) Your nearest-neighbor classifier is trained using 25 training samples, taken at integer coordinates for $-2 \leq x_1, x_2 \leq 2$. Fortunately, your training data are correctly labeled, using the labeling function shown in Eq. (1). Thus the complete training dataset is

$$X = \left[\begin{array}{cccccccc} -2 & -2 & \ldots & 0 & 0 & 0 & \ldots & 2 \\ -2 & -1 & \ldots & 0 & 1 & 2 & \ldots & 2 \end{array}\right], \quad Y = [0, 0, \ldots, 1, 1, 0, \ldots, 0]$$

Using these 25 training examples, you construct a nearest-neighbor classifier. Draw the decision boundary of the resulting nearest-neighbor classifier.

(b) Suppose now that $f_{\vec{X}|Y}(\vec{x}|0)$ and $f_{\vec{X}|Y}(\vec{x}|1)$ are both zero-mean Gaussian pdfs, with the covariance matrices Σ_0 and Σ_1 respectfully, where

$$\Sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \Sigma_1 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

Define η to be the odds ratio, $\eta = p_Y(0)/p_Y(1)$. Find a value of η such that a Bayesian classifier gives exactly the decision boundary shown in Eq. (1) on the previous page.