• This is a CLOSED BOOK exam.

• There are a total of 100 points in the exam. Each problem specifies its point total. Plan your work accordingly.

• You must SHOW YOUR WORK to get full credit.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Score</th>
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Name: ___________________________
Useful Angles

<table>
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<tr>
<th>θ</th>
<th>cos θ</th>
<th>sin θ</th>
<th>$e^{jθ}$</th>
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<tr>
<td>0</td>
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<td>0</td>
<td>1</td>
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<tr>
<td>$π/6$</td>
<td>$√3/2$</td>
<td>$1/2$</td>
<td>$√3/2 + j/2$</td>
</tr>
<tr>
<td>$π/4$</td>
<td>$√2/2$</td>
<td>$√2/2$</td>
<td>$2√2/2 + j√2/2$</td>
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<tr>
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<td>$1/2 + j√3/2$</td>
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<td>$j$</td>
</tr>
<tr>
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<td>-1</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>$3π/2$</td>
<td>1</td>
<td>-1</td>
<td>$-j$</td>
</tr>
<tr>
<td>$2π$</td>
<td>1</td>
<td>0</td>
<td>1</td>
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Gaussian Probability Densities (to Two Significant Figures)

<table>
<thead>
<tr>
<th>x</th>
<th>$\frac{1}{\sqrt{2π}} e^{-x^2/2}$</th>
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<tr>
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<td>2</td>
<td>0.05</td>
</tr>
<tr>
<td>2.5</td>
<td>0.02</td>
</tr>
<tr>
<td>3</td>
<td>0.00</td>
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</table>

Other Possibly Useful Formulas

$$X(e^{jω}) = \sum_{n=-\infty}^{\infty} x[n]e^{-jωn}$$

$$x[n] = \frac{1}{2π} \int_{-π}^{π} X(e^{jω})e^{jωn} dω$$

$$h[n] = \frac{\sin(ωe_n)}{πn} \leftrightarrow H(e^{jω}) = \begin{cases} 1 & |ω| < ω_c \\ 0 & \text{otherwise} \end{cases}$$

$$u[n] - u[n - N] \leftrightarrow e^{-jω(N-1)/2} \sin(ωN/2) / \sin(ω/2)$$

$$δ[n] \leftrightarrow 1$$

$$e^{jωn} \leftrightarrow 2πδ(ω - α)$$

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j2πkn/N}$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]e^{j2πkn/N}$$

$$S = \sum_{k=1}^{n} (\bar{x}_k - \bar{m})(\bar{x}_k - \bar{m})^T$$
Problem 1  (21 points)

You are given a 640x480 B/W input image, \( x[n_1, n_2] \) for integer pixel values \( 0 \leq n_1 \leq 639, \ 0 \leq n_2 \leq 479 \). You wish to interpolate the given pixel values in order to find the value of the image at location \( (500.3, 300.8) \). Specify the formula used to calculate \( x[500.3, 300.8] \) using each of the following algorithms. Be certain that your formula clearly states which pixels from the input image are used.

(a) Piece-wise constant interpolation.

\[
x[500.3, 300.8] = x[500, 301]
\]

(b) Bilinear interpolation.

\[
x[500.3, 300.8] = (0.7)(0.2)x[500, 300] + (0.7)(0.8)x[500, 301] + (0.3)(0.2)x[501, 300] + (0.3)(0.8)x[501, 301]
\]

(c) Sinc interpolation.

\[
x[500.3, 300.8] = \sum_{n_1=0}^{639} \sum_{n_2=0}^{479} x[n_1, n_2] \text{sinc}(\pi(500.3 - n_1)) \text{sinc}(\pi(300.8 - n_2))
\]
Problem 2  (24 points)

The images \( y[\vec{\eta}] \) and \( x[\vec{m}] \) are related by an affine transformation, where \( \vec{\eta} = [\eta, \xi, 1]^T \) and \( \vec{m} = [m, n, 1] \) are coordinate vectors of the input and output image, respectively, \( m \) is the row index, and \( n \) is the column index.

(a) The affine transformation \( \vec{\eta} = A\vec{m} \) is a rotation by \(-\frac{\pi}{3}\) radians. Find \( A \).

\[
A = \begin{bmatrix}
\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\
\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

(b) The affine transformation \( \vec{\eta} = B\vec{m} \) consists of scaling the height of the image \( m \) by a factor of 5, while keeping the width \( n \) unchanged. Find \( B \).

\[
B = \begin{bmatrix}
5 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

(c) The affine transformation \( \vec{\eta} = C\vec{m} \) consists of shifting all pixels to the left (negative \( n \) direction) by 20 columns. Find \( C \).

\[
C = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & -20 \\
0 & 0 & 1
\end{bmatrix}
\]

(d) The affine transformation \( \vec{\eta} = D\vec{m} \) consists of performing parts (a) through (c) of this problem, one after the other, in order. Specify the matrix \( D \) in terms of the matrices \( A \), \( B \), and \( C \). **There should be no numbers in your answer to this part.**

\[
D = CBA
\]
Problem 3  (11 points)

A particular triangle has corner coordinates at

\[ \vec{x}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \vec{x}_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \]

Let \( \vec{\lambda}_0 = [\lambda_1, \lambda_2, \lambda_3]^T \) be the barycentric coordinate vector corresponding to pixel \( \vec{x}_0 = \left[ \frac{2}{3}, \frac{1}{3} \right]^T \). Find \( \vec{\lambda}_0 \).

\[ \vec{\lambda}_0 = \begin{bmatrix} 0 \\ 1/3 \\ 2/3 \end{bmatrix} \]

Problem 4  (12 points)

The images \( y[\vec{\eta}] \) and \( x[\vec{m}] \) are related by an affine transformation \( \vec{\eta} = A\vec{m} \), where \( \vec{\eta} = [\eta, \xi, 1]^T \) and \( \vec{m} = [m, n, 1]^T \) are coordinate vectors of the input and output images, respectively. It is known that under this transformation, the origin swaps places with the point \([2, 2]\), thus

\[ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 2 \\ 2 \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]

Specify the \( A \) matrix as completely as you can. There should be two scalar variables in your answer; you may use the variables names \( \alpha \) and \( \beta \).

\[ A = \begin{bmatrix} \alpha & -1 - \alpha & 2 \\ \beta & -1 - \beta & 2 \\ 0 & 0 & 1 \end{bmatrix} \]
Problem 5  (16 points)

You are creating a recommender system that tries to recommend songs that will be considered to be similar to a given query. Each song is characterized by a two-dimensional vector $\vec{x}_k = [b_k, v_k]^T$ where $b_k$ is the number of beats per minute, and $v_k$ is the fraction of air-time during which there is a human voice. Your customer considers the following four songs to be similar:

$$[\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4] = \begin{bmatrix} 120 & 140 & 140 & 120 \\ 0.3 & 0.3 & 0.5 & 0.5 \end{bmatrix}$$

You are given two more test data, $\vec{x}_5 = [b_5, v_5]^T$ and $\vec{x}_6 = [b_6, v_6]^T$, and you are asked whether or not $\vec{x}_5$ and $\vec{x}_6$ should be considered similar. Write formulas for the Mahalanobis distance between $\vec{x}_5$ and $\vec{x}_6$ under the following conditions:

(a) Estimate a diagonal data covariance matrix directly from the data, and use it to write the squared Mahalanobis distance $d^2_{\Sigma}(\vec{x}_5, \vec{x}_6)$.

$$d^2_{\Sigma}(\vec{x}_5, \vec{x}_6) = \frac{(b_5 - b_6)^2}{100} + \frac{(v_5 - v_6)^2}{0.01}$$

(b) Estimate a diagonal data covariance matrix from the data, then regularize it using regularization parameter $\lambda = 0.01$ before using the result to write the squared Mahalanobis distance $d^2_{\Sigma}(\vec{x}_5, \vec{x}_6)$.

$$d^2_{\Sigma}(\vec{x}_5, \vec{x}_6) = \frac{(b_5 - b_6)^2}{100.01} + \frac{(v_5 - v_6)^2}{0.02}$$
Problem 6  (16 points)

A particular 6 megapixel image contains 3 million red pixels \((r, g, b) = (255, 0, 0)\) and 3 million blue pixels \((r, g, b) = (0, 0, 255)\).

Define its 8-quantile color histogram \(h[k_R, k_G]\) to be an \(8 \times 8\) table of numbers, specifying the number of pixels having redshift in the \(k_R\)th quantile (where smaller \(k_R\) indicates smaller redshift, \(0 \leq k_R \leq 7\)), and greenshift in the \(k_G\)th quantile (\(0 \leq k_G \leq 7\)).

(a) Find \(h[k_R, k_G]\).

\[
h[k_R, k_G] = \begin{cases} 
3 \times 10^6 & k_R = 7, k_G = 0 \\
3 \times 10^6 & k_R = 0, k_G = 0 \\
0 & \text{otherwise}
\end{cases}
\]

(b) Suppose that there is another 6 megapixel image with 3 million black pixels \((r, g, b) = (1, 1, 1)\) and 3 million white pixels \((r, g, b) = (255, 255, 255)\). Say that the color histogram of this image is called \(g[k_R, k_G]\). What is \(\|g[k_R, k_G] - h[k_R, k_G]\|\), the distance between the color histogram of the black-white image and the color histogram of the red-blue image?

\[
g[k_R, k_G] = \begin{cases} 
6 \times 10^6 & k_R = 2, k_G = 2 \\
0 & \text{otherwise}
\end{cases}
\]

Correct answer can be either the \(\ell_1\) or \(\ell_2\) norm:

\[
\|g[k_R, k_G] - h[k_R, k_G]\|_2 = 3\sqrt{6} \times 10^6 \\
\|g[k_R, k_G] - h[k_R, k_G]\|_1 = 12 \times 10^6
\]