Problem 1  (25 points)

The stock market alternates between long bull markets (state 1) and short bear markets (state 2). This HMM has the following parameters:

\[ \pi = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0.999 & 0.001 \\ 0.005 & 0.995 \end{bmatrix}, \quad \mu_1 = 0.1, \quad \mu_2 = -0.3, \quad \sigma_1^2 = \sigma_2^2 = 1, \]

where \( \pi_i = p(q_t = i) \), \( a_{ij} = p(q_{t+1} = j | q_t = i) \), and \( p(x_t | q_t = j) = \mathcal{N}(x_t; \mu_j, \sigma^2_j) \).

You observe \( x_2 \) on day 2.

For what values of \( x_2 \) does the forward algorithm yield probabilities \( \alpha_t(i) \) such that \( \alpha_2(2) > \alpha_2(1) \)?

Asking exactly the same question in different words: for what values of \( x_2 \) would it be rational to conclude that a bear market has started?

**SOLUTION:** \( x_2 < -0.1 - 2.5 \ln(999) \)

Problem 2  (15 points)

A triangle begins at

\[ X = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \]

Notice that, in this triangle, the barycentric coordinates of any point \((x_4, y_4)\) are given by

\[ \lambda_4 = \begin{bmatrix} 1 - x_4 - y_4 \\ x_4 \\ y_4 \end{bmatrix} \]

Suppose the triangle is rotated, shifted and scaled to the new position

\[ \Xi = \begin{bmatrix} \xi_1 & \xi_2 & \xi_3 \\ \eta_1 & \eta_2 & \eta_3 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix} \]

The point \((x_4, y_4) = \left(\frac{1}{3}, \frac{1}{3}\right)\), internal to the first triangle, gets mapped to some point \((\xi_4, \eta_4)\). Find \( \xi_4 \) and \( \eta_4 \).
SOLUTION: $\xi_4 = \frac{1}{3}, \ \eta_4 = 2$

Problem 3  (10 points)

Suppose a particular image has the following pixel values:

\[ a[0,0] = 1, \ a[1,0] = 0, \ a[0,1] = 0, \ a[1,1] = 0 \]

Use bilinear interpolation to estimate the value of the pixel \( a \left( \frac{1}{3}, \frac{1}{3} \right) \).

SOLUTION: \( a \left( \frac{1}{3}, \frac{1}{3} \right) = \frac{4}{9} \)

Problem 4  (25 points)

Consider two PDFs. Class \( y = 0 \) is Gaussian:

\[ p(x|y = 0) = \mathcal{N}(x; \mu_0, \sigma_0^2) \]

Class \( y = 1 \) is mixture Gaussian, and for some reason, one of its mixture components is the Gaussian from class 0:

\[ p(x|y = 1) = 0.9p(x|y = 0) + 0.1\mathcal{N}(x; \mu_1, \sigma_1^2) \]

where \( \mu_0 = 0, \ \mu_1 = 3, \ \text{and} \ \sigma_0^2 = \sigma_1^2 = 1. \)

For what values of \( x \) is

\[ \frac{p(x|y = 1)}{p(x|y = 0)} > 1? \]

SOLUTION: \( x > \frac{3}{2} \)

Problem 5  (25 points)

You have six audio training examples:

\[ \vec{a}_1 = \begin{bmatrix} 400 \\ 0 \end{bmatrix}, \ \vec{a}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ \vec{a}_3 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \ \vec{a}_4 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \ \vec{a}_5 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \ \vec{a}_6 = \begin{bmatrix} -400 \\ 0 \end{bmatrix} \]

You also have six video training examples:

\[ v_1 = -5, \ v_2 = -3, \ v_3 = -1, \ v_4 = 1, \ v_5 = 3, \ v_6 = 5 \]

The class labels for these audiovisual training data are as follows:

\[ y_1 = 0, \ y_2 = 0, \ y_3 = 0, \ y_4 = 1, \ y_5 = 1, \ y_6 = 1 \]

Define the classification function \( f(\vec{a}, v, y) = \lambda \ln p(y|\vec{a}) + (1 - \lambda) \ln p(y|v) \), where
• $p(y|\bar{a})$ is estimated using a \textbf{3-nearest neighbor (3NN)} pmf,

• $p(y|v)$ is estimated using a \textbf{5-nearest neighbor (5NN)} pmf.

Suppose $v_7 = 2$, and $\bar{a}_7 = [0.5, 0.5]$. For what values of $\lambda$ is $f(\bar{a}_7, v_7, 1) > f(\bar{a}_7, v_7, 0)$?

\textbf{SOLUTION:} $\lambda < $ \frac{\ln(3/2)}{\ln(3)}