Lecture 12: Impulse Response

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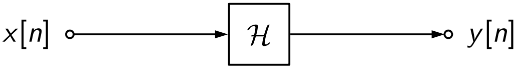
Outline

What is a System?

A **system** is anything that takes one signal as input, and generates another signal as output. We can write

$$x[n] \xrightarrow{\mathcal{H}} y[n]$$

which means



Linearity and Shift Invariance

- A system is **linear** if and only if, for any two inputs $x_1[n]$ and $x_2[n]$ that produce outputs $y_1[n]$ and $y_2[n]$,

$$x[n] = x_1[n] + x_2[n] \xrightarrow{\mathcal{H}} y[n] = y_1[n] + y_2[n]$$

- A system is **shift-invariant** if and only if, for any input $x_1[n]$ that produces output $y_1[n]$,

$$x[n] = x_1[n - n_0] \xrightarrow{\mathcal{H}} y[n] = y_1[n - n_0]$$

Outline

LSI Systems and Convolution

We care about linearity and shift-invariance because of the following remarkable result:

LSI Systems and Convolution

Let $\mathcal{H}$ be any system,

$$x[n] \xrightarrow{H} y[n]$$

If $\mathcal{H}$ is linear and shift-invariant, then whatever processes it performs can be equivalently replaced by a convolution:

$$y[n] = \sum_{m=-\infty}^{\infty} h[m]x[n-m]$$

Impulse Response

$$y[n] = \sum_{m=-\infty}^{\infty} h[m]x[n-m]$$

The weights $h[m]$ are called the "impulse response" of the system. We can measure them, in the real world, by putting the following signal into the system:

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases}$$

and measuring the response:

$$\delta[n] \xrightarrow{H} h[n]$$

Convolution: Proof

1. $h[n]$ is the impulse response.

$$\delta[n] \xrightarrow{H} h[n]$$

2. The system is **shift-invariant**, therefore

$$\delta[n-m] \xrightarrow{H} h[n-m]$$

3. The system is **linear**, therefore **scaling the input by a constant** results in **scaling the output by the same constant**:

$$x[m]\delta[n-m] \xrightarrow{H} x[m]h[n-m]$$

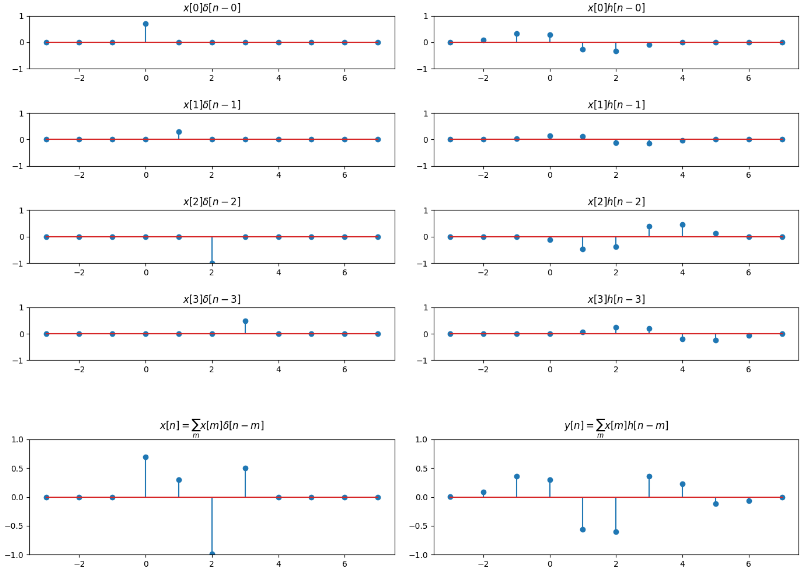
4. The system is **linear**, therefore **adding input signals** results in **adding the output signals**:

$$\sum_{m=-\infty}^{\infty} x[m]\delta[n-m] \xrightarrow{H} \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

Convolution: Proof (in Words)

- The input signal, $x[n]$, is just a bunch of samples.
- Each one of those samples is a scaled impulse, so each one of them produces a scaled impulse response at the output.
- Convolution = add together those scaled impulse responses.

Convolution: Proof (in Pictures)



Outline

Written Example

Consider a system that computes the summation of all of its inputs:

$$y[n] = \sum_{m=-\infty}^{n} x[m]$$

What is the impulse response of this system? Show that this system can be implemented using $y[n] = h[n] * x[n]$ for an appropriate $h[n]$.

Outline

Summary

- A system is **linear** if and only if, for any two inputs $x_1[n]$ and $x_2[n]$ that produce outputs $y_1[n]$ and $y_2[n]$,

$$x[n] = x_1[n] + x_2[n] \xrightarrow{\mathcal{H}} y[n] = y_1[n] + y_2[n]$$

- A system is **shift-invariant** if and only if, for any input $x_1[n]$ that produces output $y_1[n]$,

$$x[n] = x_1[n - n_0] \xrightarrow{\mathcal{H}} y[n] = y_1[n - n_0]$$

- If a system is **linear and shift-invariant** (LSI), then it can be implemented using convolution:

$$y[n] = h[n] * x[n]$$

where $h[n]$ is the impulse response:

$$\delta[n] \xrightarrow{\mathcal{H}} h[n]$$