

Lecture 24: Block Diagrams and the Inverse Z Transform

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ECE 401: Signal and Image Analysis

- 1 Review: FIR and IIR Filters, and System Functions
- 2 The System Function and Block Diagrams
- 3 Inverse Z Transform
- 4 Summary
- 5 Written Example

Outline

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FIR and IIR Filters

- An autoregressive filter is also called **infinite impulse response (IIR)**, because $h[n]$ has infinite length.
- A filter with only feedforward coefficients, and no feedback coefficients, is called **finite impulse response (FIR)**, because $h[n]$ has finite length (its length is just the number of feedforward terms in the difference equation).

System Functions

A first-order autoregressive filter,

$$y[n] = x[n] + bx[n - 1] + ay[n - 1],$$

has the impulse response and system function

$$h[n] = a^n u[n] + ba^{n-1} u[n - 1] \leftrightarrow H(z) = \frac{1 + bz^{-1}}{1 - az^{-1}},$$

where a is called the **pole** of the filter, and $-b$ is called its **zero**.

Causality and Stability

- A filter is **causal** if and only if the output, $y[n]$, depends only on **current and past** values of the input, $x[n], x[n - 1], x[n - 2], \dots$
- A filter is **stable** if and only if **every** finite-valued input generates a finite-valued output. A causal first-order IIR filter is stable if and only if $|a| < 1$.

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Why use block diagrams?

A first-order difference equation looks like

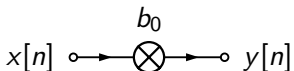
$$y[n] = b_0x[n] + b_1x[n - 1] + ay[n - 1]$$

- It's pretty easy to understand what computation is taking place in a first-order difference equation.
- As we get to higher-order systems, though, the equations for implementing them will be kind of complicated.
- In order to make the complicated equations very easy, we represent the equations using block diagrams.

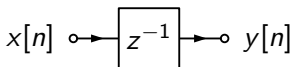
Elements of a block diagram

A block diagram has just three main element types:

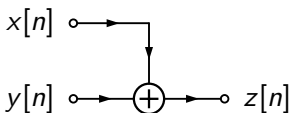
- ① **Multiplier:** the following element means $y[n] = b_0x[n]$:



- ② **Unit Delay:** the following element means $y[n] = x[n - 1]$ (i.e., $Y(z) = z^{-1}X(z)$):

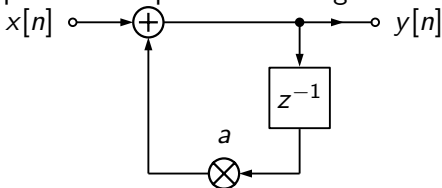


- ③ **Adder:** the following element means $z[n] = x[n] + y[n]$:



Example: Time Domain

Here's an example of a complete block diagram:



This block diagram is equivalent to the following equation:

$$y[n] = x[n] + ay[n - 1]$$

Notice that we can read it, also, as

$$Y(z) = X(z) + az^{-1}Y(z) \quad \Rightarrow \quad H(z) = \frac{1}{1 - az^{-1}}$$

A Complete First-Order IIR Filter

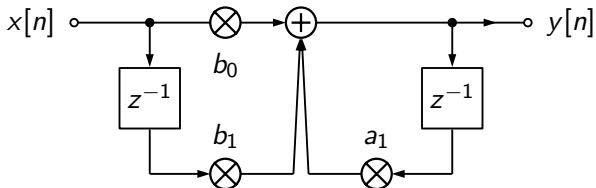
Now consider how we can represent a complete first-order IIR filter, including both the pole and the zero. Here it is in the z -domain:

$$Y(z) = b_0X(z) + b_1z^{-1}X(z) + a_1z^{-1}Y(z).$$

When we implement it, we would write a line of python that does this:

$$y[n] = b_0x[n] + b_1x[n - 1] + a_1y[n - 1],$$

which is exactly this block diagram:



Series and Parallel Combinations

Now let's talk about how to combine systems.

- **Series combination:** passing the signal through two systems **in series** is like multiplying the system functions:

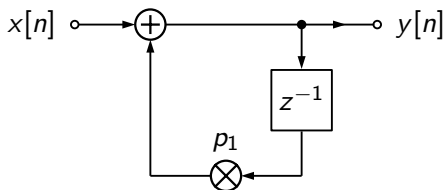
$$H(z) = H_2(z)H_1(z)$$

- **Parallel combination:** passing the signal through two systems **in parallel**, then adding the outputs, is like adding the system functions:

$$H(z) = H_1(z) + H_2(z)$$

One Block for Each System

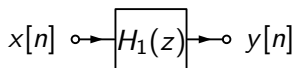
Suppose that one of the two systems, $H_1(z)$, looks like this:



and has the system function

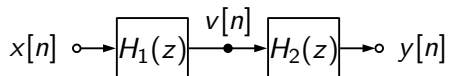
$$H_1(z) = \frac{1}{1 - p_1 z^{-1}}$$

Let's represent the whole system using a single box:



Series Combination

The series combination, then, looks like this:



This means that

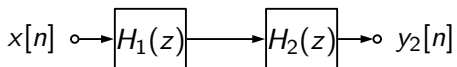
$$Y(z) = H_2(z)V(z) = H_2(z)H_1(z)X(z)$$

and therefore

$$H(z) = \frac{Y(z)}{X(z)} = H_1(z)H_2(z)$$

Series Combination

The series combination, then, looks like this:



Suppose that we know each of the systems separately:

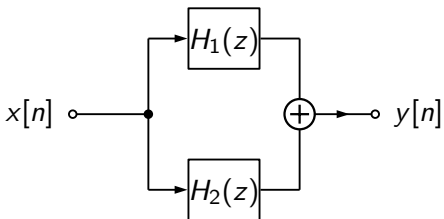
$$H_1(z) = \frac{1}{1 - p_1 z^{-1}}, \quad H_2(z) = \frac{1}{1 - p_2 z^{-1}}$$

Then, to get $H(z)$, we just have to multiply:

$$H(z) = \frac{1}{(1 - p_1 z^{-1})(1 - p_2 z^{-1})} = \frac{1}{1 - (p_1 + p_2)z^{-1} + p_1 p_2 z^{-2}}$$

Parallel Combination

Parallel combination of two systems looks like this:



This means that

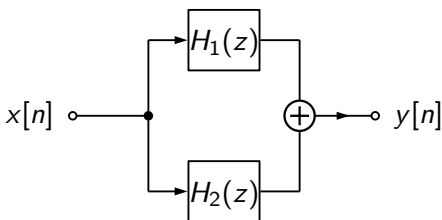
$$Y(z) = H_1(z)X(z) + H_2(z)X(z)$$

and therefore

$$H(z) = \frac{Y(z)}{X(z)} = H_1(z) + H_2(z)$$

Parallel Combination

Parallel combination of two systems looks like this:



Suppose that we know each of the systems separately:

$$H_1(z) = \frac{1}{1 - p_1 z^{-1}}, \quad H_2(z) = \frac{1}{1 - p_2 z^{-1}}$$

Then, to get $H(z)$, we just have to add:

$$H(z) = \frac{1}{1 - p_1 z^{-1}} + \frac{1}{1 - p_2 z^{-1}} = \frac{2 - (p_1 + p_2)z^{-1}}{1 - (p_1 + p_2)z^{-1} + p_1 p_2 z^{-2}}$$

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Inverse Z transform

Suppose you know $H(z)$, and you want to find $h[n]$. How can you do that?

How to find the inverse Z transform

Any IIR filter $H(z)$ can be written as...

- **denominator terms**, each with this form:

$$G_\ell(z) = \frac{1}{1 - az^{-1}} \quad \leftrightarrow \quad g_\ell[n] = a^n u[n],$$

- each possibly multiplied by a **numerator** term, like this one:

$$D_k(z) = b_k z^{-k} \quad \leftrightarrow \quad d_k[n] = b_k \delta[n - k].$$

Step #1: Numerator Terms

Consider one that you already know:

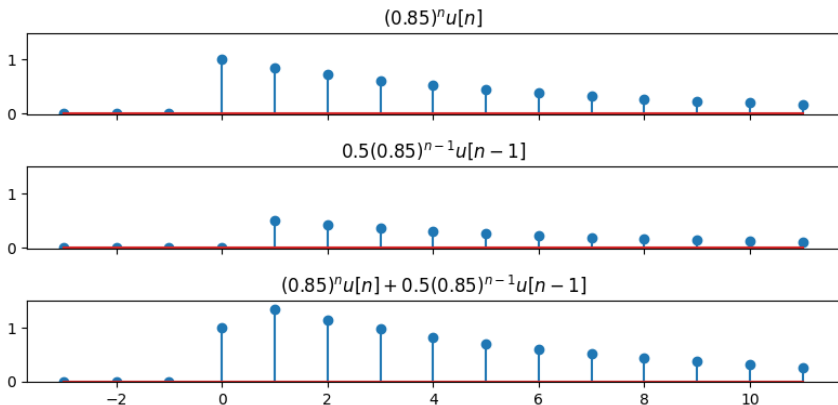
$$H(z) = \frac{1 + bz^{-1}}{1 - az^{-1}} = \left(\frac{1}{1 - az^{-1}} \right) + bz^{-1} \left(\frac{1}{1 - az^{-1}} \right)$$

and therefore

$$h[n] = (a^n u[n]) + b (a^{n-1} u[n-1])$$

Step #1: Numerator Terms

So here is the inverse transform of $H(z) = \frac{1+0.5z^{-1}}{1-0.85z^{-1}}$:



Step #1: Numerator Terms

In general, if

$$G(z) = \frac{1}{A(z)}$$

for any polynomial $A(z)$, and

$$H(z) = \frac{\sum_{k=0}^M b_k z^{-k}}{A(z)}$$

then

$$h[n] = b_0 g[n] + b_1 g[n-1] + \cdots + b_M g[n-M]$$

Step #2: Denominator Terms

Now we need to figure out the inverse transform of

$$G(z) = \frac{1}{A(z)}$$

We will solve this using a method called **partial fraction expansion**.

Step #2: Partial Fraction Expansion

Partial fraction expansion works like this:

- 1 Factor $A(z)$:

$$G(z) = \frac{1}{\prod_{\ell=1}^N (1 - p_{\ell}z^{-1})}$$

- 2 Assume that $G(z)$ is the result of a parallel system combination:

$$G(z) = \frac{C_1}{1 - p_1z^{-1}} + \frac{C_2}{1 - p_2z^{-1}} + \dots$$

- 3 Find the constants, C_{ℓ} , that make the equation true. Such constants always exist, as long as none of the roots are repeated ($p_k \neq p_{\ell}$ for $k \neq \ell$).

Partial Fraction Expansion: Example

Step # 1: Factor it:

$$\frac{1}{1 - 1.2z^{-1} + 0.72z^{-2}} = \frac{1}{(1 - (0.6 + j0.6)z^{-1})(1 - (0.6 - j0.6)z^{-1})}$$

Step #2: Express it as a sum:

$$\frac{1}{1 - 1.2z^{-1} + 0.72z^{-2}} = \frac{C_1}{1 - (0.6 + j0.6)z^{-1}} + \frac{C_2}{1 - (0.6 - j0.6)z^{-1}}$$

Step #3: Find the constants. The algebra is annoying, but it turns out that:

$$C_1 = \frac{1}{2} - j\frac{1}{2}, \quad C_2 = \frac{1}{2} + j\frac{1}{2}$$

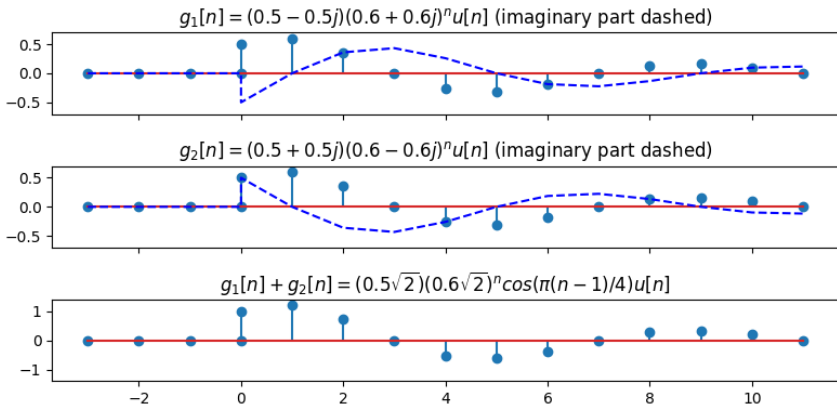
Partial Fraction Expansion: Example

The system function is:

$$\begin{aligned} G(z) &= \frac{1}{1 - 1.2z^{-1} + 0.72z^{-2}} \\ &= \frac{0.5 - 0.5j}{1 - (0.6 + j0.6)z^{-1}} + \frac{0.5 + 0.5j}{1 - (0.6 - j0.6)z^{-1}} \end{aligned}$$

and therefore the impulse response is:

$$\begin{aligned} g[n] &= (0.5 - 0.5j)(0.6 + 0.6j)^n u[n] + (0.5 + 0.5j)(0.6 - j0.6)^n u[n] \\ &= \left(0.5\sqrt{2}e^{-j\frac{\pi}{4}} \left(0.6\sqrt{2}e^{j\frac{\pi}{4}} \right)^n + 0.5\sqrt{2}e^{j\frac{\pi}{4}} \left(0.6\sqrt{2}e^{-j\frac{\pi}{4}} \right)^n \right) u[n] \\ &= \sqrt{2}(0.6\sqrt{2})^n \cos\left(\frac{\pi}{4}(n-1)\right) u[n] \end{aligned}$$



How to find the inverse Z transform

Any IIR filter $H(z)$ can be written as . . .

- a **partial fraction expansion** into a sum of **denominator** terms, each with this form:

$$G_\ell(z) = \frac{1}{1 - az^{-1}} \quad \leftrightarrow \quad g_\ell[n] = a^n u[n],$$

- each possibly multiplied by a **numerator** term, like this one:

$$D_k(z) = b_k z^{-k} \quad \leftrightarrow \quad d_k[n] = b_k \delta[n - k].$$

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Summary: Block Diagrams

- A **block diagram** shows the delays, additions, and multiplications necessary to compute output from input.
- **Series combination**: passing the signal through two systems **in series** is like multiplying the system functions:

$$H(z) = H_2(z)H_1(z)$$

- **Parallel combination**: passing the signal through two systems in **parallel**, then adding the outputs, is like adding the system functions:

$$H(z) = H_1(z) + H_2(z)$$

Summary: Inverse Z Transform

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Written Example

Find the inverse Z transform of

$$H(z) = \frac{1 - 0.7z^{-1}}{1 - 0.81z^{-2}}$$