Lecture 15: Discrete-Time Fourier Transform

Mark Hasegawa-Johnson

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Outline

Response of LSI System to Periodic Inputs

Suppose we compute $y[n] = x[n] * h[n]$, where

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j2\pi kn/N}, \text{ and}$$

$$y[n] = \frac{1}{N}\sum_{k=0}^{N-1} Y[k]e^{j2\pi kn/N}.$$

The relationship between $Y[k]$ and $X[k]$ is given by the frequency response:

$$Y[k] = H(k\omega_0)X[k]$$

where

$$H(\omega) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}$$

Response of LSI System to Aperiodic Inputs

But what about signals that never repeat themselves?
Can we still write something like

$$Y(\omega) = H(\omega)X(\omega)?$$

Outline

Aperiodic

An "aperiodic signal" is a signal that is not periodic.

- Music: strings, woodwinds, and brass are periodic, drums and rain sticks are aperiodic.
- Speech: vowels and nasals are periodic, plosives and fricatives are aperiodic.
- Images: stripes are periodic, clouds are aperiodic.
- Bioelectricity: heartbeat is periodic, muscle contractions are aperiodic.

Periodic

The spectrum of a periodic signal is given by its Fourier series. In discrete time, that's:

$$X_k = \frac{1}{N_0} \sum_{n=-\frac{N_0}{2}}^{\frac{N_0-1}{2}} x[n] e^{-j\frac{2\pi kn}{N_0}}$$
$$= \frac{1}{N_0} \sum_{n=-\frac{N_0}{2}}^{\frac{N_0-1}{2}} x[n] e^{-j\omega n}$$

and that gives the frequency content of the signal, at the frequency $\omega = \frac{2\pi k}{N_0}$.

Here I'm using $n \in \left\{-\frac{N_0}{2}, \ldots, \frac{N_0-1}{2}\right\}$, but the sum could be over any sequence of N_0 continuous samples.

Aperiodic

An aperiodic signal is one that **never** repeats itself. So we want something like the limit, as $N_0 \to \infty$, of the Fourier series. Here is the simplest such thing that is useful:

Discrete-Time Fourier Transform (DTFT)

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

Fourier Series vs. Fourier Transform

The Fourier Series coefficients are:

$$X_k = \frac{1}{N_0} \sum_{n=-\frac{N_0}{2}}^{\frac{N_0-1}{2}} x[n] e^{-j\omega n}$$

The Fourier transform is:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

Notice that, besides taking the limit as $N_0 \to \infty$, we also got rid of the $\frac{1}{N_0}$ factor. So we can think of the DTFT as

$$X(\omega) = \lim_{N_0 \to \infty, \omega = \frac{2\pi k}{N_0}} N_0 X_k$$

where the limit is: as $N_0 \to \infty$, and $k \to \infty$, but $\omega = \frac{2\pi k}{N_0}$ remains constant.

Inverse DTFT

In order to convert $X(\omega)$ back to $x[n]$, we'll take advantage of orthogonality:

$$\int_{-\pi}^{\pi} e^{j\omega(m-n)} d\omega = \begin{cases} 2\pi & m = n \\ 0 & (m-n) = \text{any nonzero integer} \end{cases}$$

Inverse DTFT

Taking advantage of orthogonality, we can see that

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega m}d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}\right)e^{j\omega m}d\omega \\
&= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} x[n]\int_{-\pi}^{\pi} e^{j\omega(m-n)}d\omega \\
= x[m]
\end{aligned}$$

Fourier Series and Fourier Transform

Discrete-Time Fourier Series (DTFS):

$$X_k = \frac{1}{N_0} \sum_{n=0}^{N_0-1} x[n] e^{-j\frac{2\pi kn}{N_0}}$$

$$x[n] = \sum_{k=0}^{N_0-1} X_k e^{j\frac{2\pi kn}{N_0}}$$

Discrete-Time Fourier Transform (DTFT):

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$$

Outline

Properties of the DTFT

In order to better understand the DTFT, let's discuss these properties:

0. Periodicity
1. Linearity
2. Time Shift
3. Frequency Shift
4. Filtering is Convolution

Property #4 is actually the reason why we invented the DTFT in the first place. Before we discuss it, though, let's talk about the others.

0. Periodicity

The DTFT is periodic with a period of 2π. That's just because $e^{j2\pi}=1$:

$$X(\omega)=\sum_n x[n]e^{-j\omega n}$$

$$X(\omega+2\pi)=\sum_n x[n]e^{-j(\omega+2\pi)n}=\sum_n x[n]e^{-j\omega n}=X(\omega)$$

$$X(\omega-2\pi)=\sum_n x[n]e^{-j(\omega-2\pi)n}=\sum_n x[n]e^{-j\omega n}=X(\omega)$$

For example, the inverse DTFT can be defined in two different ways:

$$x[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi}X(\omega)e^{j\omega n}d\omega=\frac{1}{2\pi}\int_{0}^{2\pi}X(\omega)e^{j\omega n}d\omega$$

Those two integrals are equal because $X(\omega+2\pi)=X(\omega)$.

1. Linearity

The DTFT is linear:

$$z[n] = ax[n] + by[n] \quad \leftrightarrow \quad Z(\omega) = aX(\omega) + bY(\omega)$$

Proof:

$$\begin{aligned} Z(\omega) &= \sum_n z[n]e^{-j\omega n} \\ &= a\sum_n x[n]e^{-j\omega n} + b\sum_n y[n]e^{-j\omega n} \\ &= aX(\omega) + bY(\omega) \end{aligned}$$

2. Time Shift Property

Shifting in time is the same as multiplying by a complex exponential in frequency:

$$z[n] = x[n - n_0] \quad \leftrightarrow \quad Z(\omega) = e^{-j\omega n_0} X(\omega)$$

Proof:

$$\begin{aligned} Z(\omega) &= \sum_{n=-\infty}^{\infty} x[n - n_0] e^{-j\omega n} \\ &= \sum_{m=-\infty}^{\infty} x[m] e^{-j\omega(m+n_0)} \quad \text{(where } m = n - n_0\text{)} \\ &= e^{-j\omega n_0} X(\omega) \end{aligned}$$

3. Frequency Shift Property

Shifting in frequency is the same as multiplying by a complex exponential in time:

$$z[n] = x[n]e^{j\omega_0 n} \quad \leftrightarrow \quad Z(\omega) = X(\omega - \omega_0)$$

Proof:

$$\begin{aligned} Z(\omega) &= \sum_{n=-\infty}^{\infty} x[n]e^{j\omega_0 n}e^{-j\omega n} \\ &= \sum_{n=-\infty}^{\infty} x[n]e^{-j(\omega-\omega_0)n} \\ &= X(\omega - \omega_0) \end{aligned}$$

4. Convolution Property

Convolving in time is the same as multiplying in frequency:

$$y[n] = h[n] * x[n] \quad \leftrightarrow Y(\omega) = H(\omega)X(\omega)$$

Proof: Remember that $y[n] = h[n] * x[n]$ means that $y[n] = \sum_{m=-\infty}^{\infty} h[m]x[n-m]$. Therefore,

$$Y(\omega) = \sum_{n=-\infty}^{\infty} \left(\sum_{m=-\infty}^{\infty} h[m]x[n-m] \right) e^{-j\omega n}$$

$$= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \left(h[m]x[n-m]\right) e^{-j\omega m} e^{-j\omega(n-m)}$$

$$= \left(\sum_{m=-\infty}^{\infty} h[m]e^{-j\omega m} \right) \left(\sum_{(n-m)=-\infty}^{\infty} x[n-m]e^{-j\omega(n-m)} \right)$$

$$= H(\omega)X(\omega)$$

Outline

Impulse and Delayed Impulse

For our examples today, let's consider different combinations of these three signals:

$$f[n] = \delta[n]$$
$$g[n] = \delta[n-3]$$
$$h[n] = \delta[n-6]$$

Remember from last time what these mean:

$$f[n] = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$g[n] = \begin{cases} 1 & n = 3 \\ 0 & \text{otherwise} \end{cases}$$

$$h[n] = \begin{cases} 1 & n = 6 \\ 0 & \text{otherwise} \end{cases}$$

DTFT of an Impulse

First, let's find the DTFT of an impulse:

$$f[n] = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} F(\omega) &= \sum_{n=-\infty}^{\infty} f[n] e^{-j\omega n} \\ &= 1 \times e^{-j\omega 0} \\ &= 1 \end{aligned}$$

So we get that $f[n] = \delta[n] \leftrightarrow F(\omega) = 1$. That seems like it might be important.

DTFT of a Delayed Impulse

Second, let's find the DTFT of a delayed impulse:

$$g[n] = \begin{cases} 1 & n = 3 \\ 0 & \text{otherwise} \end{cases}$$

$$G(\omega) = \sum_{n=-\infty}^{\infty} g[n] e^{-j\omega n}$$
$$= 1 \times e^{-j\omega 3}$$

So we get that

$$g[n] = \delta[n-3] \leftrightarrow G(\omega) = e^{-j3\omega}$$

Similarly, we could show that

$$h[n] = \delta[n-6] \leftrightarrow H(\omega) = e^{-j6\omega}$$

Impulse and Delayed Impulse

So our signals are:

$$f[n] = \delta[n] \leftrightarrow F(\omega) = 1$$
$$g[n] = \delta[n-3] \leftrightarrow G(\omega) = e^{-3j\omega}$$
$$h[n] = \delta[n-6] \leftrightarrow H(\omega) = e^{-6j\omega}$$

Time Shift Property

Notice that

$$g[n] = f[n-3]$$
$$h[n] = g[n-3].$$

From the time-shift property of the DTFT, we can get that

$$G(\omega) = e^{-j3\omega}F(\omega)$$
$$H(\omega) = e^{-j3\omega}G(\omega).$$

Plugging in $F(\omega) = 1$, we get

$$G(\omega) = e^{-j3\omega}$$
$$H(\omega) = e^{-j6\omega},$$

which we already know to be the right answer!

Convolution Property and the Impulse

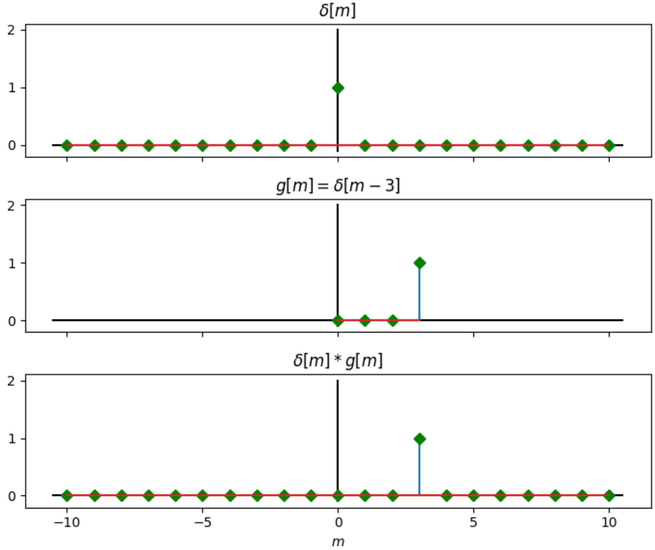
Notice that, if $F(\omega) = 1$, then anything times $F(\omega)$ gives itself again. In particular,

$$G(\omega) = G(\omega)F(\omega)$$
$$H(\omega) = H(\omega)F(\omega)$$

Since multiplication in frequency is the same as convolution in time, that must mean that when you convolve any signal with an impulse, you get the same signal back again:

$$g[n] = g[n] * \delta[n]$$
$$h[n] = h[n] * \delta[n]$$

Convolution Property and the Impulse



Convolution Property and the Delayed Impulse

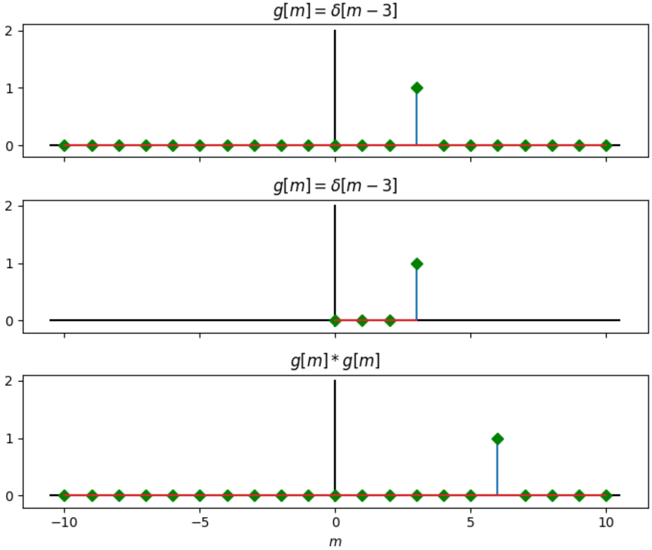
Here's another interesting thing. Notice that $G(\omega) = e^{-j3\omega}$, but $H(\omega) = e^{-j6\omega}$. So

$$\begin{aligned} H(\omega) &= e^{-j3\omega}e^{-j3\omega} \\ &= G(\omega)G(\omega) \end{aligned}$$

Does that mean that:

$$\delta[n-6] = \delta[n-3] * \delta[n-3]$$

Convolution Property and the Delayed Impulse



Outline

Summary

The DTFT (discrete time Fourier transform) of any signal is $X(\omega)$, given by

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$$

Particular useful examples include:

$$f[n] = \delta[n] \leftrightarrow F(\omega) = 1$$

$$g[n] = \delta[n-n_0] \leftrightarrow G(\omega) = e^{-j\omega n_0}$$

Properties of the DTFT

Properties worth knowing include:

0. Periodicity: $X(\omega+2\pi)=X(\omega)$
1. Linearity:

$$z[n]=ax[n]+by[n]\leftrightarrow Z(\omega)=aX(\omega)+bY(\omega)$$

2. Time Shift: $x[n-n_0]\leftrightarrow e^{-j\omega n_0}X(\omega)$
3. Frequency Shift: $e^{j\omega_0 n}x[n]\leftrightarrow X(\omega-\omega_0)$
4. Filtering is Convolution:

$$y[n]=h[n]*x[n]\leftrightarrow Y(\omega)=H(\omega)X(\omega)$$

Outline

Written Example

Suppose that $h[n]$ and $x[n]$ are identical rectangle functions:

$$x[n] = h[n] = \begin{cases} 1 & -5 \leq n \leq 5 \\ 0 & \text{otherwise} \end{cases}$$

1. Find $y[n] = h[n] * x[n]$ by calculating the convolution.
2. Find $H(\omega)$.
3. Find $Y(\omega) = H(\omega)X(\omega)$