Lecture 13: Frequency Response

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Outline

What is Signal Processing, Really?

- When we process a signal, usually, we're trying to enhance the meaningful part, and reduce the noise.
- **Spectrum** helps us to understand which part is meaningful, and which part is noise.
- **Convolution** (a.k.a. filtering) is the tool we use to perform the enhancement.
- **Frequency Response** of a filter tells us exactly which frequencies it will enhance, and which it will reduce.

Review: Convolution

- A **convolution** is exactly the same thing as a **weighted local average**. We give it a special name, because we will use it very often. It's defined as:

$$y[n] = \sum_m h[m]f[n-m] = \sum_m h[n-m]f[m]$$

- We use the symbol $*$ to mean "convolution:"

$$y[n] = h[n] * f[n] = \sum_m h[m]f[n-m] = \sum_m h[n-m]f[m]$$

Review: Spectrum

The **spectrum** of $x(t)$ is the set of frequencies, and their associated phasors,

$$\text{Spectrum}\,(x(t)) = \{(f_{-N}, a_{-N}), \ldots, (f_0, a_0), \ldots, (f_N, a_N)\}$$

such that

$$x(t) = \sum_{k=-N}^{N} a_k e^{j2\pi f_k t}$$

Review: Fourier Series

One reason the spectrum is useful is that **any** periodic signal can be written as a sum of cosines. Fourier's theorem says that any $x(t)$ that is periodic, i.e.,

$$x(t+T_0) = x(t)$$

can be written as

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi k F_0 t}$$

which is a special case of the spectrum for periodic signals: $f_k = kF_0$, and $a_k = X_k$, and

$$F_0 = \frac{1}{T_0}$$

- **Fourier Series Analysis** (finding the spectrum, given the waveform):

$$X_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j2\pi kt/T_0}dt$$

- **Fourier Series Synthesis** (finding the waveform, given the spectrum):

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi kt/T_0}$$

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Frequency Response

The **frequency response**, $H(\omega)$, of a filter $h[n]$, is its output in response to a pure tone, expressed as a function of the frequency of the tone.

Calculating the Frequency Response

- **Output of the filter:**

$$y[n] = h[n] * x[n] = \sum_m h[m]x[n-m]$$

- **in response to a pure tone:**

$$x[n] = e^{j\omega n}$$

Calculating the Frequency Response

Output of the filter in response to a pure tone:

$$
\begin{aligned}
y[n] &= \sum_m h[m]x[n-m] \\
&= \sum_m h[m]e^{j\omega(n-m)} \\
&= e^{j\omega n}\left(\sum_m h[m]e^{-j\omega m}\right)
\end{aligned}
$$

Notice that the part inside the parentheses is not a function of n. It is not a function of m, because the m gets summed over. It is only a function of ω. It is called the frequency response of the filter, $H(\omega)$.

Frequency Response

When the input to a filter is a pure tone,

$$x[n] = e^{j\omega n},$$

then its output is the same pure tone, scaled and phase shifted by a complex number called the **frequency response** $H(\omega)$:

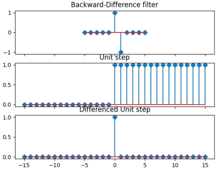
$$y[n] = H(\omega)e^{j\omega n}$$

The frequency response is related to the impulse response as

$$H(\omega) = \sum_m h[m]e^{-j\omega m}$$

Outline

Example: First Difference



Remember that taking the difference between samples can be written as a convolution:

$$y[n] = x[n] - x[n-1] = h[n] * x[n],$$

where

$$h[n] = \begin{cases} 1 & n = 0 \\ -1 & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

Example: First Difference

Suppose that the input is a pure tone:

$$x[n] = e^{j\omega n}$$

Then the output will be

$$\begin{aligned} y[n] &= x[n] - x[n-1] \\ &= e^{j\omega n} - e^{j\omega(n-1)} \\ &= H(\omega)e^{j\omega n}, \end{aligned}$$

where

$$H(\omega) = 1 - e^{-j\omega}$$

Example: First Difference

So we have some pure-tone input,

$$x[n] = e^{j\omega n}$$

...and we send it through a first-difference system:

$$y[n] = x[n] - x[n-1]$$

...and what we get, at the output, is a pure tone, scaled by $|H(\omega)|$, and phase-shifted by $\angle H(\omega)$:

$$y[n] = H(\omega)e^{j\omega n}$$

Example: First Difference

- How much is the scaling?
- How much is the phase shift?

Let's find out.

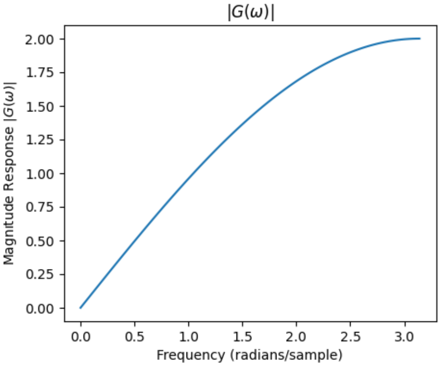
$$\begin{aligned} H(\omega) &= 1 - e^{-j\omega} \\ &= e^{-j\frac{\omega}{2}}\left(e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}\right) \\ &= e^{-j\frac{\omega}{2}}\left(2j\sin\left(\frac{\omega}{2}\right)\right) \\ &= \left(2\sin\left(\frac{\omega}{2}\right)\right)\left(e^{j\left(\frac{\pi-\omega}{2}\right)}\right) \end{aligned}$$

So the magnitude and phase response are:

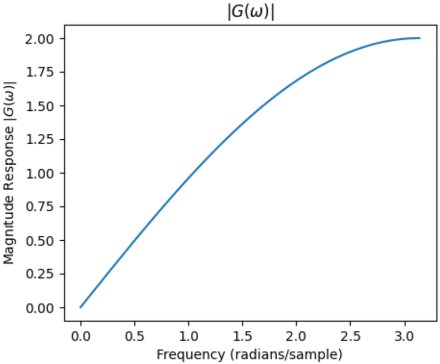
$$\begin{aligned} |H(\omega)| &= 2\sin\left(\frac{\omega}{2}\right) \\ \angle H(\omega) &= \frac{\pi-\omega}{2} \end{aligned}$$

First Difference: Magnitude Response

$$|H(\omega)| = 2\sin\left(\frac{\omega}{2}\right)$$



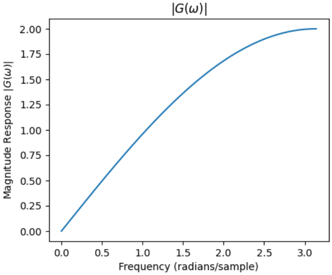
First Difference Filter at $\omega = 0$



Suppose we put in the signal $x[n] = ej\omega n$, but at the frequency $\omega = 0$. At that frequency, $x[n] = 1$. So

$$y[n] = x[n] - x[n-1] = 0$$

First Difference Filter at $\omega = \pi$



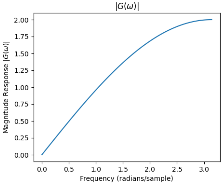
Frequency $\omega = \pi$ means the input is $(-1)^n$:

$$x[n] = e^{j\pi n} = (-1)^n = \begin{cases} 1 & n \text{ is even} \\ -1 & n \text{ is odd} \end{cases}$$

So

$$y[n] = x[n] - x[n-1] = 2x[n]$$

First Difference Filter at $\omega = \frac{\pi}{2}$



Frequency $\omega = \frac{\pi}{2}$ means the input is j^n:

$$x[n] = e^{j\frac{\pi n}{2}} = j^n$$

The frequency response is:

$$G\left(\frac{\pi}{2}\right) = 1 - e^{-j\frac{\pi}{2}} = 1 - \left(\frac{1}{j}\right),$$

The output is

$$y[n] = x[n] - x[n-1] = j^n - j^{n-1} = \left(1 - \frac{1}{j}\right) j^n$$

Outline

Superposition and the Frequency Response

The frequency response obeys the principle of **superposition**, meaning that if the input is the sum of two pure tones:

$$x[n] = e^{j\omega_1 n} + e^{j\omega_2 n},$$

then the output is the sum of the same two tones, each scaled by the corresponding frequency response:

$$y[n] = H(\omega_1)e^{j\omega_1 n} + H(\omega_2)e^{j\omega_2 n}$$

Response to a Cosine

There are no complex exponentials in the real world. Instead, we'd like to know the output in response to a cosine input. Fortunately, a cosine is the sum of two complex exponentials:

$$x[n] = \cos(\omega n) = \frac{1}{2}e^{j\omega n} + \frac{1}{2}e^{-j\omega n},$$

therefore,

$$y[n] = \frac{H(\omega)}{2}e^{j\omega n} + \frac{H(-\omega)}{2}e^{-j\omega n}$$

Response to a Cosine

What is $H(-\omega)$? Remember the definition:

$$H(\omega) = \sum_m h[m]e^{-j\omega m}$$

Replacing every ω with a $-\omega$ gives:

$$H(-\omega) = \sum_m h[m]e^{j\omega m}.$$

Notice that $h[m]$ is real-valued, so the only complex number on the RHS is $e^{j\omega m}$. But

$$e^{j\omega m} = \left(e^{-j\omega m}\right)^*$$

so

$$H(-\omega) = H^*(\omega)$$

Response to a Cosine

$$y[n] = \frac{H(\omega)}{2}e^{j\omega n} + \frac{H^*(\omega)}{2}e^{-j\omega n}$$

$$= \frac{|H(\omega)|}{2}e^{j\angle H(\omega)}e^{j\omega n} + \frac{|H(\omega)|}{2}e^{-j\angle H(\omega)}e^{-j\omega n}$$

$$= |H(\omega)|\cos\left(\omega n + \angle H(\omega)\right)$$

Response to a Cosine

If

$$x[n] = \cos(\omega n)$$

...then ...

$$y[n] = |H(\omega)| \cos\left(\omega n + \angle H(\omega)\right)$$

Magnitude and Phase Responses

- The **Magnitude Response** $|H(\omega)|$ tells you by how much a pure tone at ω will be scaled.
- The **Phase Response** $\angle H(\omega)$ tells you by how much a pure tone at ω will be advanced in phase.

Outline

Example: First Difference

Remember that the first difference, $y[n] = x[n] - x[n-1]$, is supposed to sort of approximate a derivative operator:

$$y(t) \approx \frac{d}{dt}x(t)$$

If the input is a cosine, what is the output?

$$\frac{d}{dt}\cos(\omega t) = -\omega\sin(\omega t) = \omega\cos\left(\omega t + \frac{\pi}{2}\right)$$

Does the first-difference operator behave the same way (multiply by a magnitude of $|H(\omega)| = \omega$, phase shift by $+\frac{\pi}{2}$ radians so that cosine turns into negative sine)?

Example: First Difference

Freqeuncy response of the first difference filter is

$$H(\omega) = 1 - e^{-j\omega}$$

Let's try to convert it to polar form, so we can find its magnitude and phase:

$$\begin{aligned} H(\omega) &= e^{-j\frac{\omega}{2}}\left(e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}\right) \\ &= e^{-j\frac{\omega}{2}}\left(2j\sin\left(\frac{\omega}{2}\right)\right) \\ &= \left(2\sin\left(\frac{\omega}{2}\right)\right)\left(e^{j\left(\frac{\pi-\omega}{2}\right)}\right) \end{aligned}$$

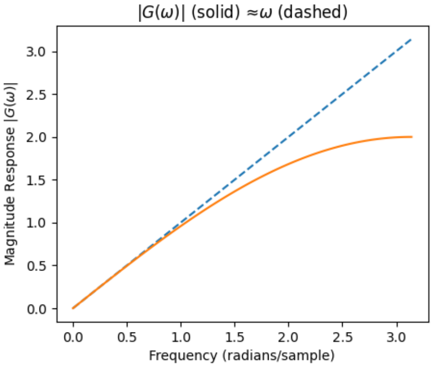
So the magnitude and phase response are:

$$|H(\omega)| = 2\sin\left(\frac{\omega}{2}\right)$$

$$\angle H(\omega) = \frac{\pi - \omega}{2}$$

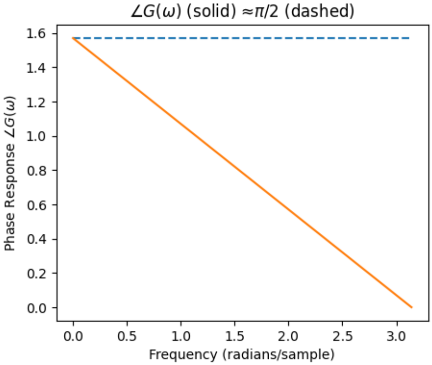
First Difference: Magnitude Response

Taking the derivative of a cosine scales it by ω. The first-difference filter scales it by $|H(\omega)| = 2\sin(\omega/2)$, which is almost the same, but not quite:



First Difference: Phase Response

Taking the derivative of a cosine shifts it, in phase, by $+\frac{\pi}{2}$ radians, so that the cosine turns into a negative sine. The first-difference filter shifts it by $\angle H(\omega) = \frac{\pi-\omega}{2}$, which is the same at very low frequencies, but very different at high frequencies.



First Difference: All Together

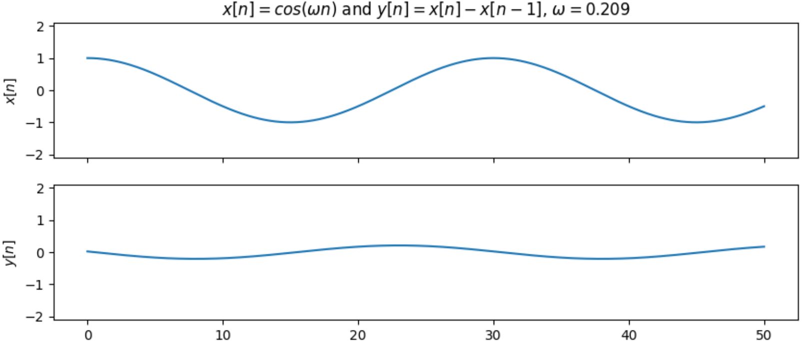
Putting it all together, if the input is $x[n] = \cos(\omega n)$, the output is

$$y[n] = |H(\omega)| \cos\left(\omega n + \angle H(\omega)\right) = 2\sin\left(\frac{\omega}{2}\right)\cos\left(\omega n + \frac{\pi - \omega}{2}\right)$$

First Difference: All Together

$$y[n] = 2\sin\left(\frac{\omega}{2}\right)\cos\left(\omega n + \frac{\pi-\omega}{2}\right)$$

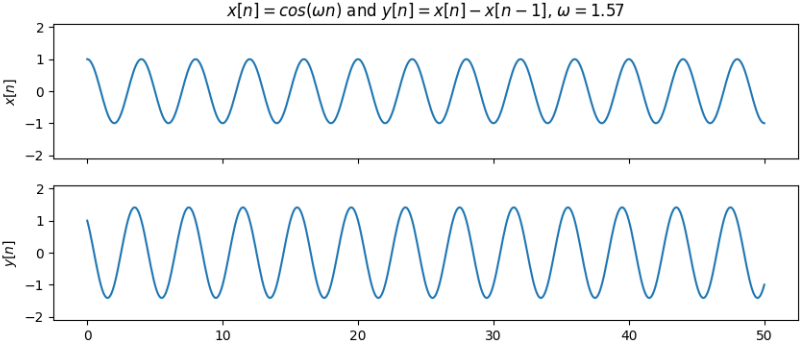
At very low frequencies, the output is almost $-\sin(\omega n)$, but with very low amplitude:



First Difference: All Together

$$y[n] = 2\sin\left(\frac{\omega}{2}\right)\cos\left(\omega n + \frac{\pi - \omega}{2}\right)$$

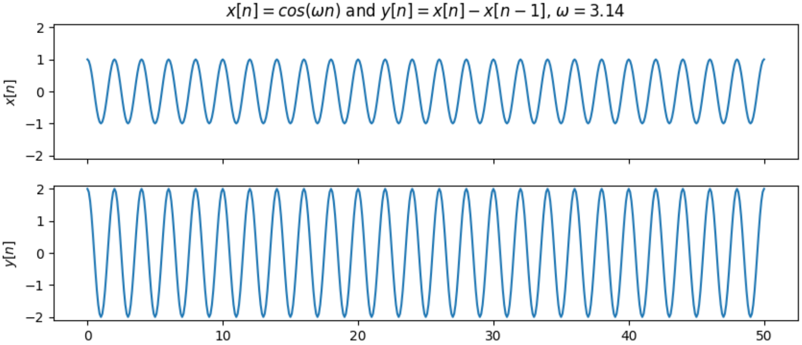
At intermediate frequencies, the phase shift between the input and output is reduced:



First Difference: All Together

$$y[n] = 2\sin\left(\frac{\omega}{2}\right)\cos\left(\omega n + \frac{\pi-\omega}{2}\right)$$

At very high frequencies, the phase shift between input and output is eliminated – the output is a cosine, just like the input:



Outline

Written Example

Consider the following system:

$$y[n] = x[n - n_0]$$

This can be written as a convolution, with impulse response

$$h[n] = \delta[n - n_0]$$

What is $H(\omega)$? Using $H(\omega)$, can you find the output of this system if $x[n] = \cos(\omega n)$?

Outline

Summary

- **Tones in → Tones out**

$$x[n] = e^{j\omega n} \rightarrow y[n] = H(\omega)e^{j\omega n}$$
$$x[n] = \cos(\omega n) \rightarrow y[n] = |H(\omega)| \cos(\omega n + \angle H(\omega))$$
$$x[n] = A\cos(\omega n + \theta) \rightarrow y[n] = A|H(\omega)| \cos(\omega n + \theta + \angle H(\omega))$$

- where the **Frequency Response** is given by

$$H(\omega) = \sum_m h[m]e^{-j\omega m}$$