Lecture 9: Exam 1 Review

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ECE 401: Signal and Image Analysis, Fall 2021

Outline

1. Topics Covered
2. Phasors
3. Spectrum
4. Fourier Series
5. Sampling and Interpolation
6. Summary

Topics Covered

1. HW1: Phasors
2. MP1: Spectrum
3. HW2: Fourier Series
4. MP2: Sampling

Outline

Phasors

$$
\begin{aligned}
x(t) &= A\cos\left(2\pi ft+\theta\right)\\
&= \Re\left\{ze^{j2\pi ft}\right\}\\
&= \frac{1}{2}z^*e^{-j2\pi ft}+\frac{1}{2}ze^{j2\pi ft}
\end{aligned}
$$

where

$$z = Ae^{j\theta}$$

Adding Phasors

How do you add

$$z(t) = A\cos\left(2\pi ft + \theta\right) + B\cos\left(2\pi ft + \phi\right)?$$

Answer:

$$z = (A\cos\theta + B\cos\phi) + j(A\sin\theta + B\sin\phi)$$

$$z(t) = \Re\left\{ze^{j2\pi ft}\right\}$$

Outline

Two-sided spectrum

The **spectrum** of $x(t)$ is the set of frequencies, and their associated phasors,

$$\text{Spectrum}\left(x(t)\right) = \{(f_{-N}, a_{-N}), \ldots, (f_0, a_0), \ldots, (f_N, a_N)\}$$

such that

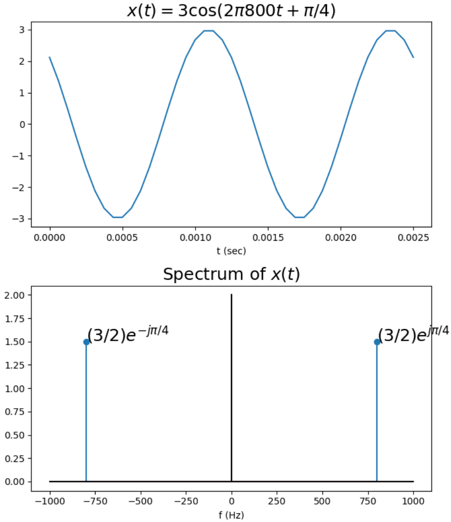
$$x(t) = \sum_{k=-N}^{N} a_k e^{j2\pi f_k t}$$

Spectrum Plots

The **spectrum plot** of a periodic signal is a plot with

- frequency on the X-axis,
- showing a vertical spike at each frequency component,
- each of which is labeled with the corresponding phasor.

Example: Cosine w/Amplitude 3, Phase $\pi/4$



Property #1: Scaling

Suppose we have a signal

$$x(t) = \sum_{k=-N}^{N} a_k e^{j2\pi f_k t}$$

Suppose we scale it by a factor of G:

$$y(t) = Gx(t)$$

That just means that we scale each of the coefficients by G:

$$y(t) = \sum_{k=-N}^{N} (Ga_k)\, e^{j2\pi f_k t}$$

Property #2: Adding a constant

Suppose we have a signal

$$x(t) = \sum_{k=-N}^{N} a_k e^{j2\pi f_k t}$$

Suppose we add a constant, C:

$$y(t) = x(t) + C$$

That just means that we add that constant to a_0:

$$y(t) = (a_0 + C) + \sum_{k\neq 0} a_k e^{j2\pi f_k t}$$

Property #3: Adding two signals

Suppose we have two signals:

$$x(t) = \sum_{n=-N}^{N} a'_n e^{j2\pi f'_n t}$$

$$y(t) = \sum_{m=-M}^{M} a''_m e^{j2\pi f''_m t}$$

and we add them together:

$$z(t) = x(t) + y(t) = \sum_k a_k e^{j2\pi f_k t}$$

where, if a frequency f_k comes from both $x(t)$ and $y(t)$, then we have to do phasor addition:

$$\text{If } f_k = f'_n = f''_m \text{ then } a_k = a'_n + a''_m$$

Property #4: Time shift

Suppose we have a signal

$$x(t) = \sum_{k=-N}^{N} a_k e^{j2\pi f_k t}$$

and we want to time shift it by τ seconds:

$$y(t) = x(t-\tau)$$

Time shift corresponds to a **phase shift** of each spectral component:

$$y(t) = \sum_{k=-N}^{N} \left(a_k e^{-j2\pi f_k \tau}\right) e^{j2\pi f_k t}$$

Property #5: Frequency shift

Suppose we have a signal

$$x(t) = \sum_{k=-N}^{N} a_k e^{j2\pi f_k t}$$

and we want to shift it in frequency by some constant overall shift, F:

$$y(t) = \sum_{k=-N}^{N} a_k e^{j2\pi (f_k+F)t}$$

Frequency shift corresponds to amplitude modulation (multiplying it by a complex exponential at the carrier frequency F):

$$y(t) = x(t)e^{j2\pi Ft}$$

Property #6: Differentiation

Suppose we have a signal

$$x(t) = \sum_{k=-N}^{N} a_k e^{j2\pi f_k t}$$

and we want to differentiate it:

$$y(t) = \frac{dx}{dt}$$

Differentiation corresponds to scaling each spectral coefficient by $j2\pi f_k$:

$$y(t) = \sum_{k=-N}^{N} \left(j2\pi f_k a_k\right) e^{j2\pi f_k t}$$

Outline

Fourier Series

- **Analysis** (finding the spectrum, given the signal):

$$X_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j2\pi kt/T_0}dt$$

- **Synthesis** (finding the signal, given the spectrum):

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi kt/T_0}$$

Discrete-Time Fourier Series

- **Analysis** (finding the spectrum, given the signal):

$$X_k = \frac{1}{N_0} \sum_{0}^{N_0-1} x[n] e^{-j2\pi kn/N_0}$$

- **Synthesis** (finding the signal, given the spectrum):

$$x[n] = \sum_{k} X_k e^{j2\pi kn/N_0}$$

where the sum is over any set of N_0 consecutive harmonics.

Spectral Properties of Fourier Series

- **Scaling:**

$$y(t) = Gx(t) \Leftrightarrow Y_k = GX_k$$

- **Add a Constant:**

$$y(t) = x(t) + C \Leftrightarrow Y_k = \begin{cases} X_0 + C & k = 0 \\ X_k & \text{otherwise} \end{cases}$$

- **Add Signals:** Suppose that $x(t)$ and $y(t)$ have the same fundamental frequency, then

$$z(t) = x(t) + y(t) \Leftrightarrow Z_k = X_k + Y_k$$

Spectral Properties of Fourier Series

- **Time Shift:** Shifting to the right, in time, by τ seconds:

$$y(t) = x(t-\tau) \Leftrightarrow Y_k = a_k e^{-j2\pi k F_0 \tau}$$

- **Frequency Shift:** Shifting upward in frequency by F Hertz:

$$y(t) = x(t) e^{j2\pi d F_0 t} \Leftrightarrow Y_k = X_{k-d}$$

- **Differentiation:**

$$y(t) = \frac{dx}{dt} \Leftrightarrow Y_k = j2\pi k F_0 X_k$$

Outline

How to sample a continuous-time signal

Suppose you have some continuous-time signal, $x(t)$, and you'd like to sample it, in order to store the sample values in a computer. The samples are collected once every $T_s = \frac{1}{F_s}$ seconds:

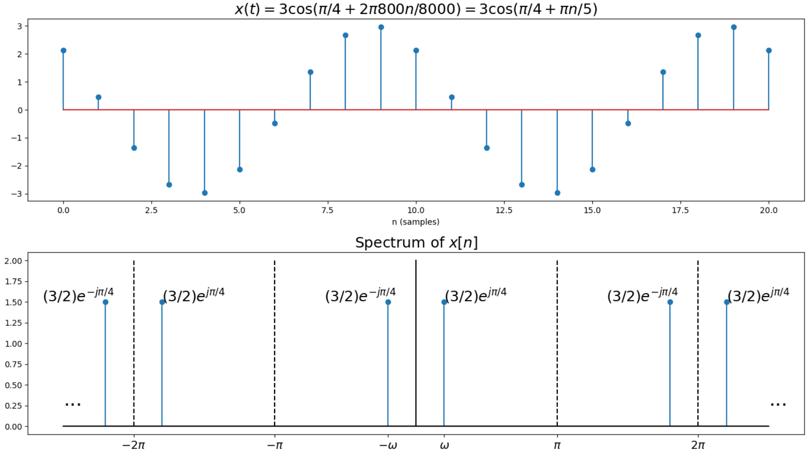
$$x[n] = x(t = nT_s)$$

Spectrum Plot of a Discrete-Time Periodic Signal

The spectrum plot of a **discrete-time periodic signal** is a regular spectrum plot, but with the X-axis relabeled. Instead of frequency in Hertz= $\left[\frac{\text{cycles}}{\text{second}}\right]$, we use

$$\omega\left[\frac{\text{radians}}{\text{sample}}\right] = \frac{2\pi\left[\frac{\text{radians}}{\text{cycle}}\right] f\left[\frac{\text{cycles}}{\text{second}}\right]}{F_s\left[\frac{\text{samples}}{\text{second}}\right]}$$

Example: Cosine w/Amplitude 3, Phase $\pi/4$



Aliasing

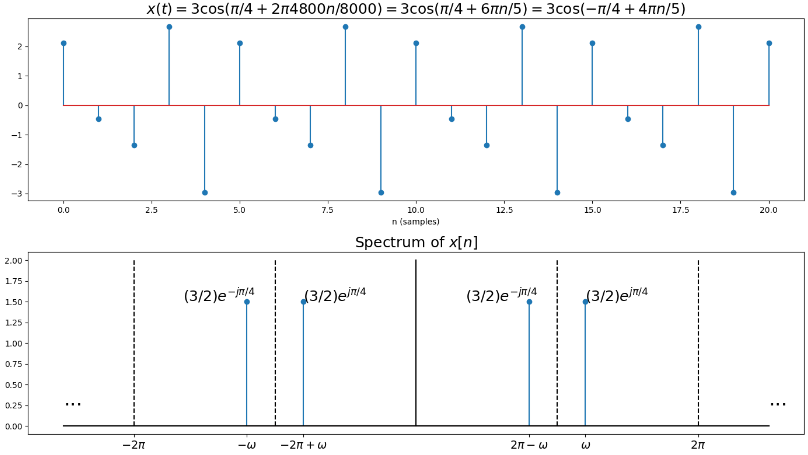
- A sampled sinusoid can be reconstructed perfectly if the Nyquist criterion is met, $f < \frac{F_s}{2}$.
- If the Nyquist criterion is violated, then:
  - If $\frac{F_s}{2} < f < F_s$, then it will be aliased to

$$f_a = F_s - f$$
$$z_a = z^*$$

    i.e., the sign of all sines will be reversed.
  - If $F_s < f < \frac{3F_s}{2}$, then it will be aliased to

$$f_a = f - F_s$$
$$z_a = z$$

Example: Cosine w/Amplitude 3, Phase $\pi/4$



Interpolation

Interpolation is the general method for reconstructing a continuous-time signal from its samples. The formula is:

$$y(t) = \sum_{n=-\infty}^{\infty} y[n]p(t - nT_s)$$

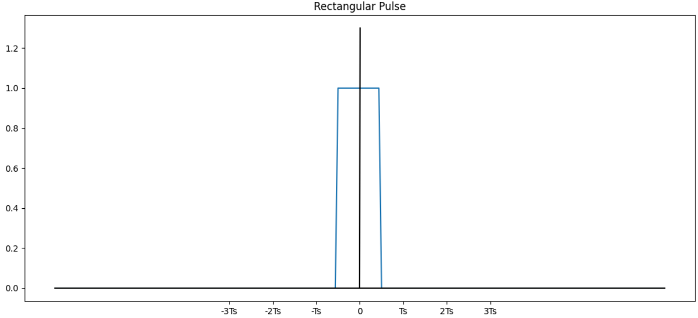
Interpolation kernels

- Piece-wise constant interpolation = interpolate using a rectangle
- Piece-wise linear interpolation = interpolate using a triangle
- Ideal interpolation = interpolate using a sinc

Rectangular pulses

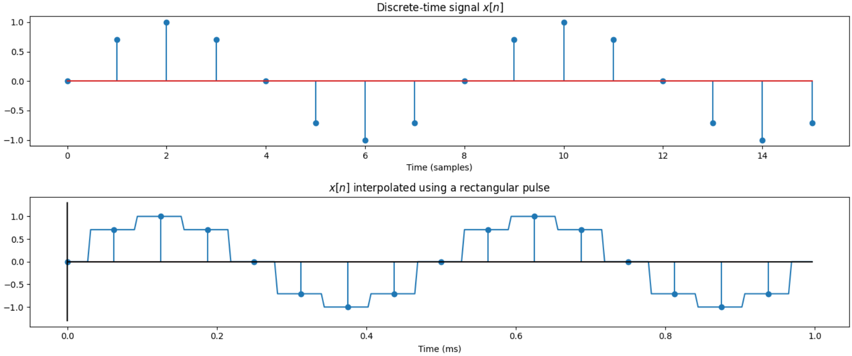
For example, suppose that the pulse is just a rectangle,

$$p(t) = \begin{cases} 1 & -\frac{T_S}{2} \leq t < \frac{T_S}{2} \\ 0 & \text{otherwise} \end{cases}$$



Rectangular pulses = Piece-wise constant interpolation

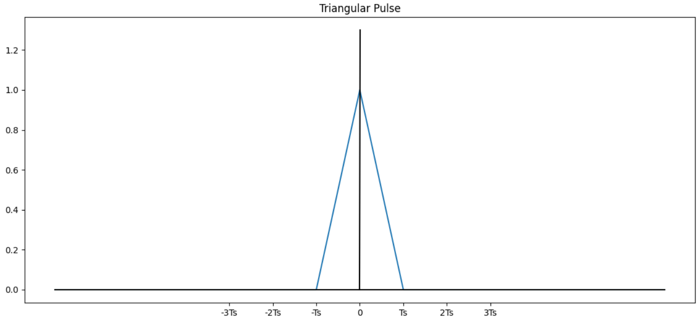
The result is a piece-wise constant interpolation of the digital signal:



Triangular pulses

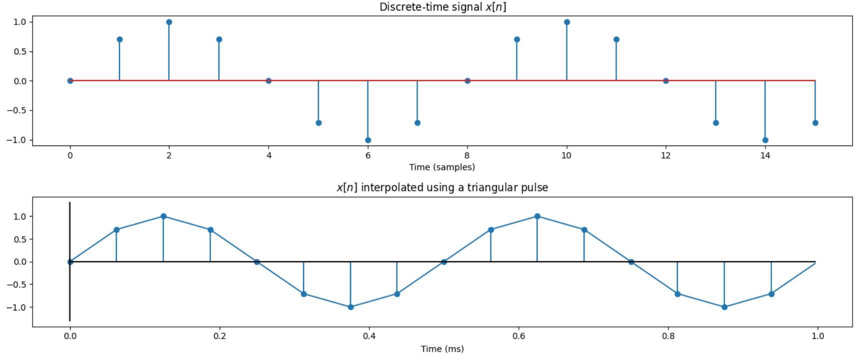
The rectangular pulse has the disadvantage that $y(t)$ is discontinuous. We can eliminate the discontinuities by using a triangular pulse:

$$p(t) = \begin{cases} 1 - \frac{|t|}{T_S} & -T_S \leq t < T_S \\ 0 & \text{otherwise} \end{cases}$$



Triangular pulses = Piece-wise linear interpolation

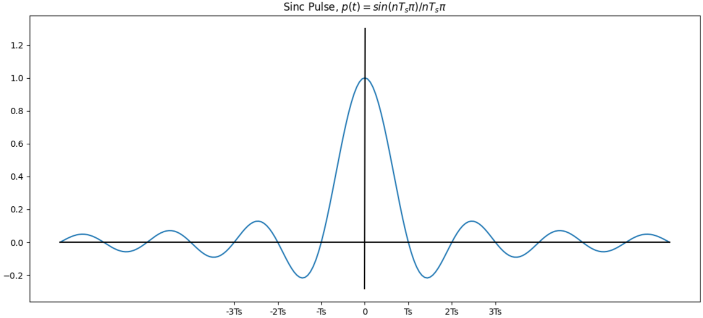
The result is a piece-wise linear interpolation of the digital signal:



Sinc pulses

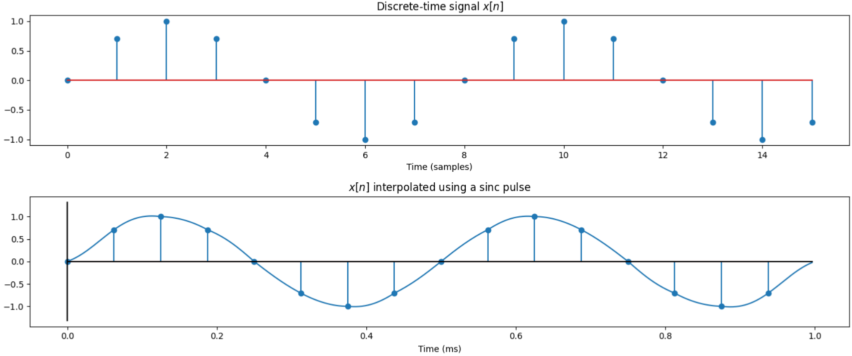
If a signal has all its energy at frequencies below Nyquist ($f < \frac{F_s}{2}$), then it can be perfectly reconstructed using sinc interpolation:

$$p(t) = \frac{\sin(\pi t/T_S)}{\pi t/T_S}$$



Sinc pulse = ideal bandlimited interpolation

If a signal has all its energy at frequencies below Nyquist ($f < \frac{F_s}{2}$), then it can be perfectly reconstructed using sinc interpolation:



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Summary: Topics Covered

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