Lecture 6: Sampling and Aliasing

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ECE 401: Signal and Image Analysis, Fall 2021

Outline

Two-sided spectrum

The **spectrum** of $x(t)$ is the set of frequencies, and their associated phasors,

$$\text{Spectrum}\,(x(t)) = \{(f_{-N}, a_{-N}), \ldots, (f_0, a_0), \ldots, (f_N, a_N)\}$$

such that

$$x(t) = \sum_{k=-N}^{N} a_k e^{j2\pi f_k t}$$

Fourier's theorem

One reason the spectrum is useful is that **any** periodic signal can be written as a sum of cosines. Fourier's theorem says that any $x(t)$ that is periodic, i.e.,

$$x(t+T_0) = x(t)$$

can be written as

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi k F_0 t}$$

which is a special case of the spectrum for periodic signals: $f_k = kF_0$, and $a_k = X_k$, and

$$F_0 = \frac{1}{T_0}$$

Fourier Series

- **Analysis** (finding the spectrum, given the waveform):

$$X_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j2\pi kt/T_0}dt$$

- **Synthesis** (finding the waveform, given the spectrum):

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi kt/T_0}$$

Outline

How to sample a continuous-time signal

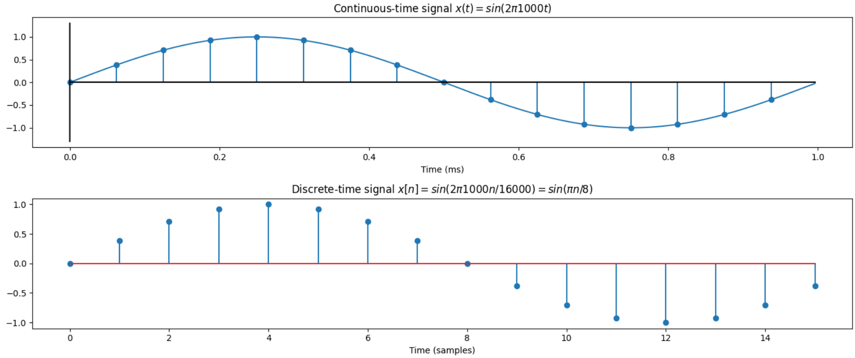
Suppose you have some continuous-time signal, $x(t)$, and you'd like to sample it, in order to store the sample values in a computer. The samples are collected once every $T_s = \frac{1}{F_s}$ seconds:

$$x[n] = x(t = nT_s)$$

Example: a 1kHz sine wave

For example, suppose $x(t) = \sin(2\pi 1000t)$. By sampling at $F_s = 16000$ samples/second, we get

$$x[n] = \sin\left(2\pi 1000 \frac{n}{16000}\right) = \sin(\pi n/8)$$



Outline

Can every sine wave be reconstructed from its samples?

The question immediately arises: can every sine wave be reconstructed from its samples?

The answer, unfortunately, is "no."

Can every sine wave be reconstructed from its samples?

For example, two signals $x_1(t)$ and $x_2(t)$, at 10kHz and 6kHz respectively:

$$x_1(t) = \cos(2\pi 10000 t), \quad x_2(t) = \cos(2\pi 6000 t)$$

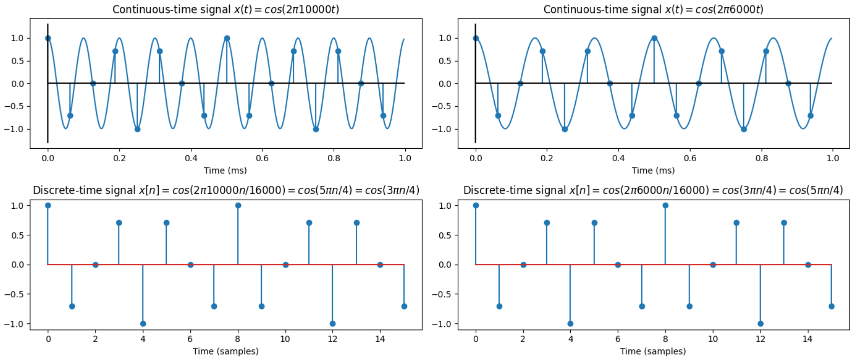
Let's sample them at $F_s = 16,000$ samples/second:

$$x_1[n] = \cos\left(2\pi 10000 \frac{n}{16000}\right), \quad x_2[n] = \cos\left(2\pi 6000 \frac{n}{16000}\right)$$

Simplifying a bit, we discover that $x_1[n] = x_2[n]$. We say that the 10kHz tone has been "aliased" to 6kHz:

$$x_1[n] = \cos\left(\frac{5\pi n}{4}\right) = \cos\left(\frac{3\pi n}{4}\right)$$
$$x_2[n] = \cos\left(\frac{3\pi n}{4}\right) = \cos\left(\frac{5\pi n}{4}\right)$$

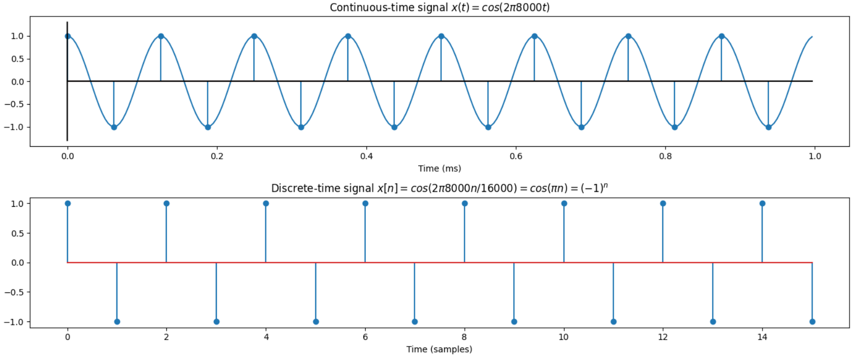
Can every sine wave be reconstructed from its samples?



What is the highest frequency that can be reconstructed?

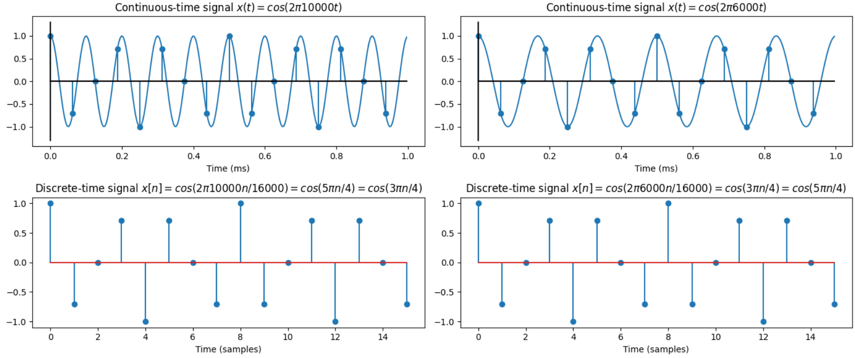
The highest frequency whose cosine can be exactly reconstructed from its samples is called the "Nyquist frequency," $F_N = F_S/2$. If $x(t) = \cos(2\pi F_N t)$, then

$$x[n] = \cos\left(2\pi F_N \frac{n}{F_S}\right) = \cos(\pi n) = (-1)^n$$



Sampling above Nyquist ⇒ Aliasing to a frequency below Nyquist

If you try to sample a signal whose frequency is above Nyquist (like the one shown on the left), then it gets **aliased** to a frequency below Nyquist (like the one shown on the right).



Outline

Aliased Frequency

Suppose you have a cosine at frequency f:

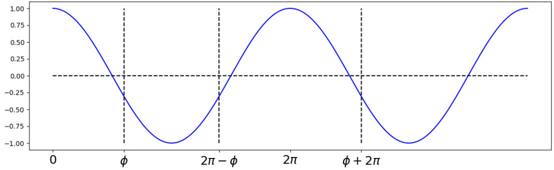
$$x(t) = \cos(2\pi f t)$$

Suppose you sample it at F_s samples/second. If F_s is not high enough, it might get aliased to some other frequency, f_a.

$$x[n] = \cos(2\pi f n/F_s) = \cos(2\pi f_a n/F_s)$$

How can you predict what f_a will be?

Aliased Frequency

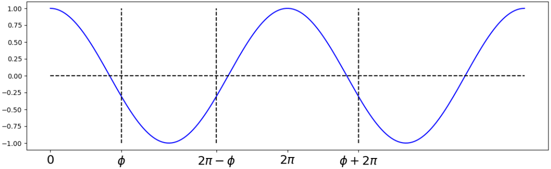


Aliasing comes from two sources:

$$\cos(\phi) = \cos(2\pi n - \phi)$$
$$\cos(\phi) = \cos(\phi - 2\pi n)$$

The equations above are true for any integer n.

Aliased Frequency



Let's plug in $\phi = \frac{2\pi f n}{F_s}$, and $2\pi = \frac{2\pi F_s}{F_s}$. That gives us:

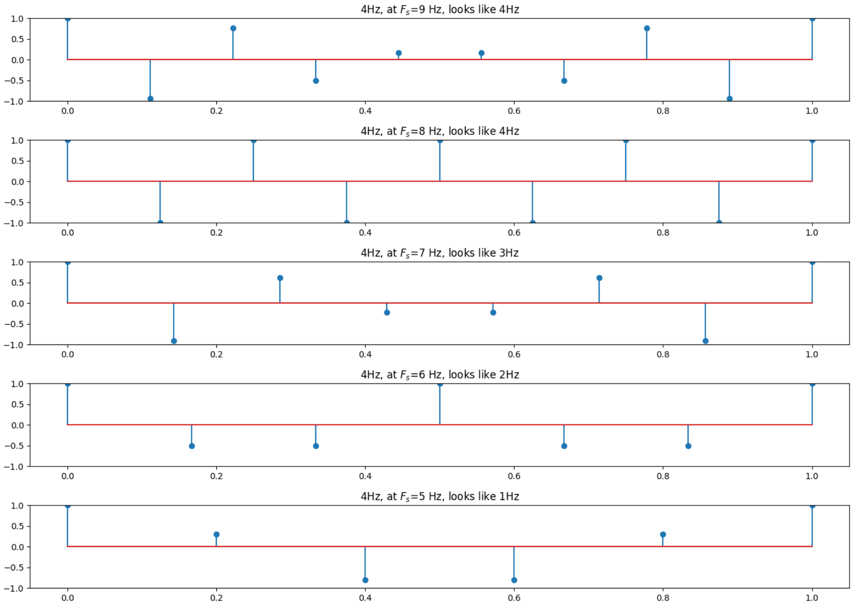
$$\cos\left(\frac{2\pi f n}{F_s}\right) = \cos\left(\frac{2\pi n(F_s - f)}{F_s}\right)$$
$$\cos\left(\frac{2\pi f n}{F_s}\right) = \cos\left(\frac{2\pi (f - F_s) n}{F_s}\right)$$

So a discrete-time cosine at frequency f is also a cosine at frequency $F_s - f$, and it's also a cosine at $f - F_s$.

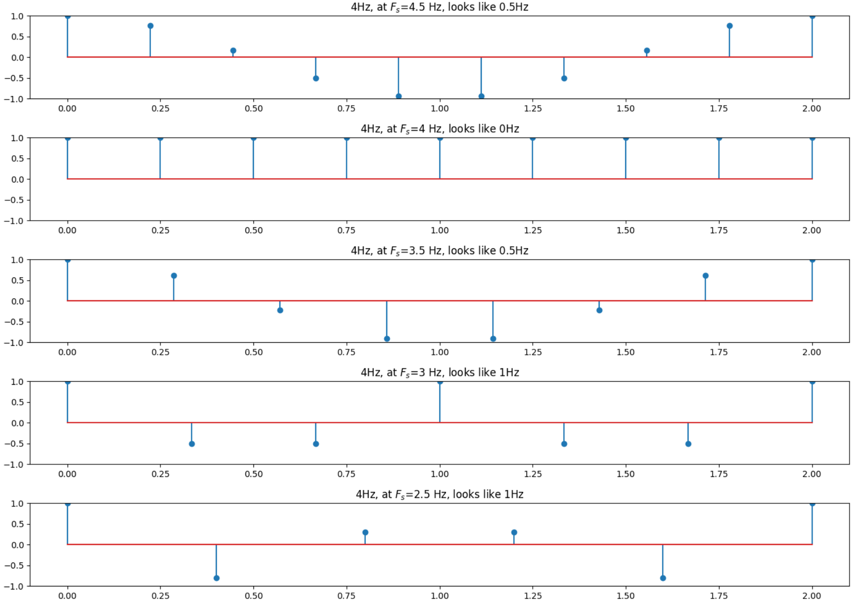
Aliased Frequency

- A discrete-time cosine at frequency f is also a cosine at frequency $F_s - f$, and it's also a cosine at $f - F_s$.
- So which of those frequencies will we hear when we play the sinusoid back again?
- **ANSWER:** any frequency that can be reconstructed by the analog-to-digital converter. That means any frequency below the Nyquist frequency, $F_N = F_s/2$.

Aliased Frequency



Aliased Frequency



Aliased Frequency

All of the following frequencies are actually **the same frequency** when a cosine is sampled at F_s samples/second.

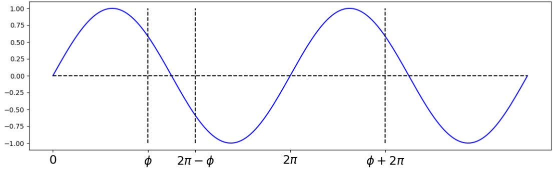
$$f_a \in \{f - \ell F_s, \ell F_s - f : \ell \in \text{any integer}\}$$

The "aliased frequency" is whichever of those is below Nyquist ($F_s/2$). Usually there's only one that's below Nyquist, so you can just look for

$$f_a = \min\left(f - \ell F_s, \ell F_s - f : \ell \in \text{any integer}\right)$$

Outline

Sine is Different

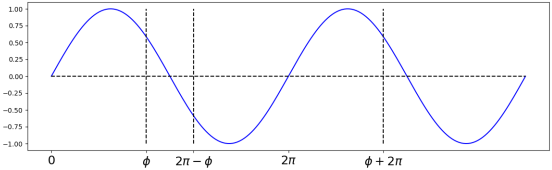


Sine waves are different for the following reason:

$$\sin(\phi) = -\sin(2\pi n - \phi)$$

$$\sin(\phi) = \sin(\phi - 2\pi n)$$

Sine is Different

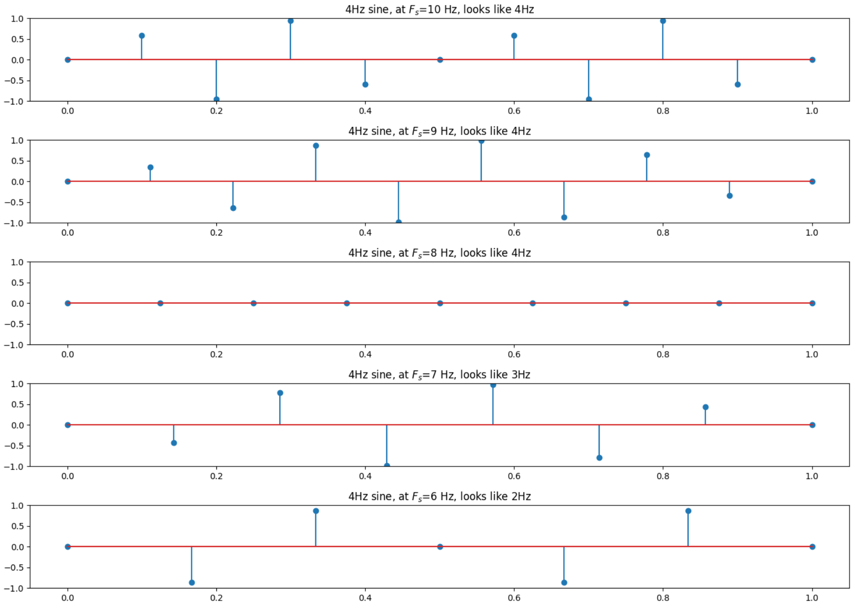


Therefore:

$$\sin\left(\frac{2\pi fn}{F_s}\right) = -\sin\left(\frac{2\pi n(F_s - f)}{F_s}\right)$$
$$\sin\left(\frac{2\pi fn}{F_s}\right) = \sin\left(\frac{2\pi (f - F_s)n}{F_s}\right)$$

So a discrete-time sine at frequency f is also a **negative** sine at frequency $F_s - f$, and a **positive** sine at frequency $f - F_s$.

Sine is Different



Aliased Phase of a General Phasor

For a general complex exponential, we get:

$$ze^{j\phi} = ze^{j(\phi-2\pi n)} = \left(z^* e^{j(2\pi n-\phi)}\right)^*$$

Therefore:

$$\Re\left\{ze^{j\frac{2\pi fn}{F_s}}\right\} = \Re\left\{ze^{j\frac{2\pi(f-F_s)n}{F_s}}\right\} = \Re\left\{z^* e^{j\frac{2\pi(F_s-f)n}{F_s}}\right\}$$

Aliased Phase of a General Phasor

Suppose we have some frequency f, and we're trying to find its aliased frequency f_a.

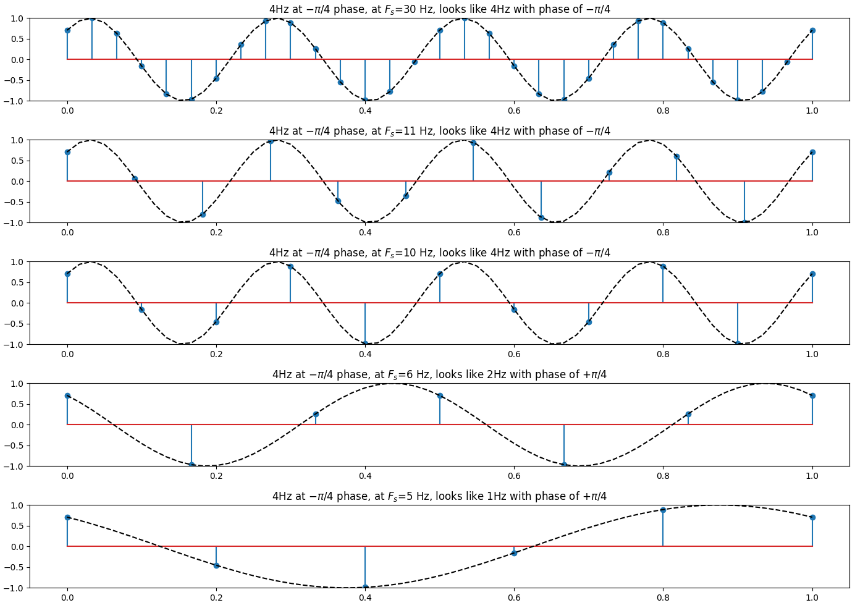
- Among the several possibilities, if $f_a = F_s - f$ is below Nyquist, then that's the frequency we'll hear. Its phasor will be the complex conjugate of the original phasor,

$$z_a = z^*$$

- On the other hand, if $f_a = f - F_s$ is below Nyquist, then that's the frequency we'll hear. Its phasor will be the same as the phasor of the original sinusoid:

$$z_a = z$$

Aliased Phase of a General Phasor



Outline

Summary

- A sampled sinusoid can be reconstructed perfectly if the Nyquist criterion is met, $f < \frac{F_s}{2}$.
- If the Nyquist criterion is violated, then:
  - If $\frac{F_s}{2} < f < F_s$, then it will be aliased to

    $$f_a = F_s - f$$
    $$z_a = z^*$$

    i.e., the sign of all sines will be reversed.
  - If $F_s < f < \frac{3F_s}{2}$, then it will be aliased to

    $$f_a = f - F_s$$
    $$z_a = z$$

Outline

Written Example

Sketch a sinusoid with some arbitrary phase (say, $-\pi/4$). Show where the samples are if it's sampled:

- more than twice per period
- more than once per period, but less than twice per period
- less than once per period