Lecture 4: Fourier Series

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ECE 401: Signal and Image Analysis, Fall 2021
1. Review: Spectrum

2. Orthogonality

3. Fourier Series

4. Example: Square Wave

5. Summary
Outline

1. Review: Spectrum
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The **spectrum** of $x(t)$ is the set of frequencies, and their associated phasors,

$$\text{Spectrum} \ (x(t)) = \{(f_{-N}, a_{-N}), \ldots, (f_0, a_0), \ldots, (f_N, a_N)\}$$

such that

$$x(t) = \sum_{k=-N}^{N} a_k e^{j2\pi f_k t}$$
Fourier’s theorem

One reason the spectrum is useful is that any periodic signal can be written as a sum of cosines. Fourier’s theorem says that any $x(t)$ that is periodic, i.e.,

$$x(t + T_0) = x(t)$$

can be written as

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi kF_0 t}$$

which is a special case of the spectrum for periodic signals: $f_k = kF_0$, and $a_k = X_k$, and

$$F_0 = \frac{1}{T_0}$$
Analysis and Synthesis

- **Fourier Synthesis** is the process of generating the signal, $x(t)$, given its spectrum. Last lecture, you learned how to do this, in general.

- **Fourier Analysis** is the process of finding the spectrum, $X_k$, given the signal $x(t)$. I’ll tell you how to do that today.
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Two functions $f(t)$ and $g(t)$ are said to be **orthogonal**, over some period of time $T$, if

$$
\int_{0}^{T} f(t)g(t) = 0
$$
Sine and Cosine are Orthogonal

For example, $\sin(2\pi t)$ and $\cos(2\pi t)$ are orthogonal over the period $0 \leq t \leq 1$:
Similarly, sinusoids at different frequencies are orthogonal over any time segment that contains an integer number of periods:
How to use orthogonality

Suppose we have a signal that is known to be

\[ x(t) = a \cos(2\pi 3t) + b \sin(2\pi 3t) + c \cos(2\pi 4t) + d \sin(2\pi 4t) + \ldots \]

\ldots but we don’t know \(a, b, c, d\), etc. Let’s use orthogonality to figure out the value of \(b\):

\[
\int_0^1 x(t) \sin(2\pi 3t) dt = a \int_0^1 \cos(2\pi 3t) \sin(2\pi 3t) dt \\
+ b \int_0^1 \sin(2\pi 3t) \sin(2\pi 3t) dt \\
+ c \int_0^1 \cos(2\pi 4t) \sin(2\pi 3t) dt \\
+ e \int_0^1 \sin(2\pi 4t) \sin(2\pi 3t) dt + \ldots
\]
How to use orthogonality

\[ \int_0^1 x(t) \sin(2\pi 3t) dt = 0 + b \int_0^1 \sin^2(2\pi 3t) dt + 0 + 0 + \ldots \]

The average value of \( \sin^2(t) \) is \( 1/2 \), so

\[ \int_0^1 x(t) \sin(2\pi 3t) dt = \frac{b}{2} \]

If we don’t know the value of \( b \), but we do know how to integrate \( \int x(t) \sin(2\pi 3t) dt \), then we can find the value of \( b \) from the formula above.
How to use orthogonality

\[ x(t) = 1.5 \sin(2\pi 3t) + 0.25 \sin(2\pi 4t) \]

\[ \int x(t) \sin(2\pi 3t) \, dt = \frac{1.5}{2} \]

\[ \int x(t) \sin(2\pi 4t) \, dt = \frac{0.25}{2} \]
How to use Orthogonality: Fourier Series

We still have one problem. Integrating $\int x(t) \cos(2\pi 4t) dt$ is hard—lots of ugly integration by parts and so on. What do we do?

1. **Fourier Series:** Instead of cosine, use complex exponential:

   $$\int x(t) e^{-j2\pi ft} dt$$

   That integral is still hard, but it’s always easier than $\int x(t) \cos(2\pi 4t) dt$. It can usually be solved with some combination of integration by parts, variable substitution, etc.
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Remember Fourier’s theorem. He said that any periodic signal, with a period of $T_0$ seconds, can be written

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi kt/T_0}$$

What I want to do, now, is to prove that if you know $x(t)$, you can find its Fourier series coefficients using the following formula:

$$X_k = \frac{1}{T_0} \int_{0}^{T_0} x(t)e^{-j2\pi kt/T_0} dt$$
Fourier’s Theorem

Remember Fourier’s theorem. He said that any periodic signal, with a period of $T_0$ seconds, can be written

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi kt/T_0}$$

I’m going to find the formula for $X_k$ using the following idea:

- **Orthogonality:** $e^{-j2\pi \ell t/T_0}$ is orthogonal to $e^{j2\pi kt/T_0}$ for any $\ell \neq k$. 
Orthogonality: start with \( x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi kt/T_0} \), and multiply both sides by \( e^{-j2\pi \ell t/T_0} \):

\[
x(t)e^{-2\pi \ell t/T_0} = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi (k-\ell)t/T_0}
\]

Now integrate both sides of that equation, over any complete period:

\[
\frac{1}{T_0} \int_{0}^{T_0} x(t)e^{-2\pi \ell t/T_0} dt = \sum_{k=-\infty}^{\infty} X_k \frac{1}{T_0} \int_{0}^{T_0} e^{j2\pi (k-\ell)t/T_0} dt
\]
Fourier’s Theorem and Orthogonality

Now think really hard about what’s inside that integral sign:

\[
\frac{1}{T_0} \int_0^{T_0} e^{j2\pi(k-\ell)t/T_0} \, dt
= \frac{1}{T_0} \int_0^{T_0} \cos \left( \frac{2\pi(k-\ell)t}{T_0} \right) \, dt \\
+ j \frac{1}{T_0} \int_0^{T_0} \sin \left( \frac{2\pi(k-\ell)t}{T_0} \right) \, dt
\]

- If \( k \neq \ell \), then we’re integrating a cosine and a sine over \( k-\ell \) periods. That integral is always zero.

- If \( k = \ell \), then we’re integrating

\[
\frac{1}{T_0} \int_0^{T_0} \cos(0) \, dt + j \frac{1}{T_0} \int_0^{T_0} \sin(0) \, dt = 1
\]
So, because of orthogonality:

\[
\frac{1}{T_0} \int_0^{T_0} x(t) e^{-2\pi \ell t/T_0} \, dt = \sum_{k=\infty}^{\infty} X_k \frac{1}{T_0} \int_0^{T_0} e^{j2\pi (k-\ell)t/T_0} \, dt = \ldots + 0 + 0 + 0 + X_\ell + 0 + 0 + 0 + \ldots
\]
**Fourier Series**

- **Analysis** (finding the spectrum, given the waveform):
  \[ X_k = \frac{1}{T_0} \int_0^{T_0} x(t)e^{-j2\pi kt/T_0} \, dt \]

- **Synthesis** (finding the waveform, given the spectrum):
  \[ x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi kt/T_0} \]
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Fourier series: Square wave example
Square wave example

Let's use a square wave with a nonzero DC value, like this one:

\[ x(t) = \begin{cases} 
1 & -\frac{T_0}{4} < t < \frac{T_0}{4} \\
0 & \text{otherwise} 
\end{cases} \]
• **Analysis** (finding the spectrum, given the waveform):

\[ X_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) e^{-j2\pi kt/T_0} dt \]
Analysis (finding the spectrum, given the waveform):

\[ X_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t)e^{-j2\pi kt/T_0} dt \]

\[ = \frac{1}{T_0} \int_{-T_0/4}^{T_0/4} e^{-j2\pi kt/T_0} dt \]
Square wave: the $X_0$ term

$$X_0 = \frac{1}{T_0} \int_{-T_0/4}^{T_0/4} e^{-j2\pi kt/T_0} dt$$

...but if $k = 0$, then $e^{-j2\pi kt/T_0} = 1$!!

$$X_0 = \frac{1}{T_0} \int_{-T_0/4}^{T_0/4} 1dt = \frac{1}{2}$$
Square wave: the $X_0$ term

$$X_0 = \frac{1}{2}$$
Square wave: the $X_k$ terms, $k \neq 0$

$$X_k = \frac{1}{T_0} \int_{-T_0/4}^{T_0/4} e^{-j2\pi kt/T_0} dt$$
Square wave: the $X_k$ terms, $k \neq 0$

$$X_k = \frac{1}{T_0} \int_{-T_0/4}^{T_0/4} e^{-j2\pi kt/T_0} \, dt$$

$$= \frac{1}{T_0} \left( \frac{1}{-j2\pi k/T_0} \right) \left[ e^{-j2\pi k T_0/T_0} \right]_{-T_0/4}^{T_0/4}$$

$$= \left( \frac{j}{2\pi k} \right) \left[ e^{-j2\pi k(T_0/4)/T_0} - e^{-j2\pi k(-T_0/4)/T_0} \right]$$

$$= \left( \frac{j}{2\pi k} \right) \left[ e^{-j\pi k/2} - e^{j\pi k/2} \right]$$

$$= \frac{1}{\pi k} \sin \left( \frac{\pi k}{2} \right)$$

$$= \begin{cases} 
0 & k \text{ even} \\
\pm \frac{1}{\pi k} & k \text{ odd} 
\end{cases}$$
Square wave: the $X_1$ terms

$$X_1 = \frac{1}{\pi}$$
Square wave: the $X_2$ term

$$X_2 = 0$$
Square wave: the $X_3$ term

$$X_3 = -\frac{1}{3\pi}$$
Square wave: the $X_5$ term

$$X_5 = \frac{1}{5\pi}$$
Square wave: the whole Fourier series

\[ x(t) = \frac{1}{2} + \frac{1}{\pi} \left( \cos \left( \frac{2\pi t}{T_0} \right) - \frac{1}{3} \cos \left( \frac{6\pi t}{T_0} \right) + \frac{1}{5} \cos \left( \frac{10\pi t}{T_0} \right) - \frac{1}{7} \ldots \right) \]
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- **Analysis** (finding the spectrum, given the waveform):
  \[ X_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j2\pi kt/T_0} dt \]

- **Synthesis** (finding the waveform, given the spectrum):
  \[ x(t) = \sum_{k=\infty}^{\infty} X_k e^{j2\pi kt/T_0} \]