Problem 5.1

Consider the signal \( x[n] = \delta[n] + \delta[n-2] \). Plot the magnitude DTFT, \( |X(\omega)| \), of this signal, for \( 0 \leq \omega < 2\pi \). Draw circles on your plot to show the frequency samples \( X[k] \) for a 4-point DFT.

Solution:

\[
X(\omega) = 1 + e^{-2j\omega} = e^{-j\omega}(e^{j\omega} + e^{-j\omega}) = e^{-j\omega}2\cos(\omega)
\]

So

\[
|X(\omega)| = 2|\cos(\omega)|
\]

The frequency samples corresponding to \( X_k \) are at the frequencies \( \omega_k = \frac{k\pi}{4} \), and have values of

\[
|X[k]| = \begin{cases} 
1 & k = 0, 2 \\
0 & k = 1, 3 
\end{cases}
\]

Problem 5.2

In this problem, we will repeat Hamming’s famous calculation, that resulted in the Hamming window. Consider a slightly modified, even-symmetric raised-cosine window,

\[
w_C[n] = \left((1 - a) + a \cos \left(\frac{2\pi n}{N}\right)\right) w_R[n]
\]

where \( a \) is an arbitrary constant, whose value has not yet been determined, and \( w_R[n] \) is

\[
w_R[n] = \begin{cases} 
1 & -M \leq n \leq M \\
0 & \text{otherwise}
\end{cases}
\]

and the total length of the window is \( N = 2M + 1 \). Recall that the DTFT of an even-symmetric rectangular window is

\[
W_R(\omega) = D_N(\omega) = \frac{\sin(\omega N/2)}{\sin(\omega/2)}
\]
(a) Use the linearity and frequency-shift properties of the DTFT to find \( W_C(\omega) \), the DTFT of \( w_C[n] \).

Solution:

\[
W_C[n] = \left( (1 - a) + a \cos \left( \frac{2\pi n}{N} \right) \right) w_R[n]
\]

\[
= (1 - a)w_R[n] + \frac{a}{2} e^{j\frac{2\pi}{N}} w_R[n] + \frac{a}{2} e^{-j\frac{2\pi}{N}} w_R[n]
\]

Using the linearity and frequency-shift properties of the DTFT, we see that the DTFT is

\[
W_C(\omega) = (1 - a)D_N(\omega) + \frac{a}{2} D_N \left( \omega - \frac{2\pi}{N} \right) + D_N \left( \omega + \frac{2\pi}{N} \right)
\]

(b) Sketch \( W_C(\omega) \), for \( 0 \leq \omega \leq \frac{10\pi}{N} \). Draw circles at the frequencies that would be sampled by an \( N \)-point DFT. Find the values of \( W_C[k] \) for all \( k \) in the range \( 0 \leq k \leq N - 1 \), as functions of \( a \) and \( N \).

Solution: The DTFT should look like a Dirichlet function, but with its main lobe twice as wide as a normal Dirichlet function. The frequency samples at \( \omega_k = \frac{2\pi k}{N} \) have the values

\[
W_C[k] = \begin{cases} 
(1 - a)N & k = 0 \\
\frac{aN}{2} & k = 1, N - 1 \\
0 & \text{otherwise}
\end{cases}
\]

(c) Find \( W_C \left( \frac{5\pi}{N} \right) \) in terms of \( a \) and \( N \), and then find the value of \( a \) that zeros it out, \( W_C \left( \frac{5\pi}{N} \right) = 0 \).

Note: in order to find the value of \( W_C \left( \frac{5\pi}{N} \right) \), you will want to take advantage of the fact that, for small enough values of \( k \),

\[
\frac{\sin(k\pi/2)}{\sin(k\pi/2N)} \approx \frac{\sin(k\pi/2)}{k\pi/2N} = \begin{cases} 
\pm \frac{2N}{k\pi} & k \text{ odd} \\
0 & k \text{ even and nonzero}
\end{cases}
\]

Solution:

\[
W_C \left( \frac{5\pi}{N} \right) = \left( \frac{(1 - a)2N}{5\pi} - \frac{a2N}{6\pi} - \frac{a2N}{14\pi} \right)
\]

In order to zero it out, we need to find the value of \( a \) such that

\[
0 = \frac{2(1 - a)}{5} - \frac{a}{3} - \frac{a}{7}
\]

Which gives us

\[
1 = a \left( 1 + \frac{5}{6} + \frac{5}{14} \right)
\]

or \( a = 0.4565217 \). If we approximate this to 2 significant figures (i.e., if we tolerate an error of up to 0.01, which is \(-40\text{dB}\) ), the approximation would be \( a \approx 0.46 \).