Lecture 13: Block Diagrams and the Inverse Z Transform

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ECE 401: Signal and Image Analysis, Fall 2020
1. Review: FIR and IIR Filters, and System Functions

2. The System Function and Block Diagrams

3. Inverse Z Transform

4. Summary
Outline

1. Review: FIR and IIR Filters, and System Functions
2. The System Function and Block Diagrams
3. Inverse Z Transform
4. Summary
An autoregressive filter is also called **infinite impulse response (IIR)**, because $h[n]$ has infinite length.

A filter with only feedforward coefficients, and no feedback coefficients, is called **finite impulse response (FIR)**, because $h[n]$ has finite length (its length is just the number of feedforward terms in the difference equation).
A first-order autoregressive filter,

\[ y[n] = x[n] + bx[n - 1] + ay[n - 1], \]

has the impulse response and transfer function

\[ h[n] = a^n u[n] + ba^{n-1} u[n - 1] \leftrightarrow H(z) = \frac{1 + bz^{-1}}{1 - az^{-1}}, \]

where \( a \) is called the pole of the filter, and \(-b\) is called its zero.
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Why use block diagrams?

A first-order difference equation looks like

$$y[n] = b_0 x[n] + b_1 x[n-1] + ay[n-1]$$

- It's pretty easy to understand what computation is taking place in a first-order difference equation.
- As we get to higher-order systems, though, the equations for implementing them will be kind of complicated.
- In order to make the complicated equations very easy, we represent the equations using block diagrams.
Elements of a block diagram

A block diagram has just three main element types:

1. **Multiplier:** the following element means $y[n] = b_0 x[n]$:

   $\begin{align*}
   y[n] &= b_0 x[n] \\
   x[n] &\rightarrow \times \rightarrow y[n]
   \end{align*}$

2. **Unit Delay:** the following element means $y[n] = x[n - 1]$ (i.e., $Y(z) = z^{-1} X(z)$):

   $\begin{align*}
   y[n] &= z^{-1} x[n] \\
   x[n] &\rightarrow z^{-1} \rightarrow y[n]
   \end{align*}$

3. **Adder:** the following element means $z[n] = x[n] + y[n]$:

   $\begin{align*}
   z[n] &= x[n] + y[n] \\
   x[n] &\rightarrow + \\
   y[n] &\rightarrow + \rightarrow z[n]
   \end{align*}$
Example: Time Domain

Here’s an example of a complete block diagram:

```
x[n] → + → y[n]
```

This block diagram is equivalent to the following equation:

\[ y[n] = x[n] + ay[n - 1] \]

Notice that we can read it, also, as

\[ Y(z) = X(z) + az^{-1}Y(z) \quad \Rightarrow \quad H(z) = \frac{1}{1 - az^{-1}} \]
Now consider how we can represent a complete first-order IIR filter, including both the pole and the zero. Here’s its system function:

\[ Y(z) = b_0 X(z) + b_1 z^{-1} X(z) + a_1 z^{-1} Y(z). \]

When we implement it, we would write a line of python that does this:

\[ y[n] = b_0 x[n] + b_1 x[n - 1] + a_1 y[n - 1], \]

which is exactly this block diagram:
Now let’s talk about how to combine systems.

- **Series combination**: passing the signal through two systems in series is like multiplying the system functions:

  \[ H(z) = H_2(z)H_1(z) \]

- **Parallel combination**: passing the signal through two systems in parallel, then adding the outputs, is like adding the system functions:

  \[ H(z) = H_1(z) + H_2(z) \]
One Block for Each System

Suppose that one of the two systems, $H_1(z)$, looks like this:

\[
x[n] \rightarrow + \rightarrow y[n]
\]

\[
x[n] \rightarrow \times \rightarrow p_1 \rightarrow z^{-1} \rightarrow y[n]
\]

and has the system function

\[
H_1(z) = \frac{1}{1 - p_1 z^{-1}}
\]

Let’s represent the whole system using a single box:

\[
x[n] \rightarrow H_1(z) \rightarrow y[n]
\]
The series combination, then, looks like this:

$$x[n] \rightarrow H_1(z) \rightarrow y_1[n] \rightarrow H_2(z) \rightarrow y_2[n]$$

This means that

$$Y_2(z) = H_2(z) Y_1(z) = H_2(z) H_1(z) X(z)$$

and therefore

$$H(z) = \frac{Y(z)}{X(z)} = H_1(z) H_2(z)$$
The series combination, then, looks like this:

\[
x[n] \xrightarrow{H_1(z)} H_2(z) \xrightarrow{} y_2[n]
\]

Suppose that we know each of the systems separately:

\[
H_1(z) = \frac{1}{1 - p_1 z^{-1}}, \quad H_2(z) = \frac{1}{1 - p_2 z^{-1}}
\]

Then, to get \(H(z)\), we just have to multiply:

\[
H(z) = \frac{1}{(1 - p_1 z^{-1})(1 - p_2 z^{-1})} = \frac{1}{1 - (p_1 + p_2) z^{-1} + p_1 p_2 z^{-2}}
\]
Parallel combination of two systems looks like this:

\[ y[n] = H_1(z) \cdot x[n] + H_2(z) \cdot x[n] \]

This means that

\[ Y(z) = H_1(z)X(z) + H_2(z)X(z) \]

and therefore

\[ H(z) = \frac{Y(z)}{X(z)} = H_1(z) + H_2(z) \]
Parallel Combination

Parallel combination of two systems looks like this:

\[
x[n] \rightarrow H_1(z) \rightarrow y[n] \quad \text{and} \quad x[n] \rightarrow H_2(z) \rightarrow y[n]
\]

Suppose that we know each of the systems separately:

\[
H_1(z) = \frac{1}{1 - p_1 z^{-1}}, \quad H_2(z) = \frac{1}{1 - p_2 z^{-1}}
\]

Then, to get \( H(z) \), we just have to add:

\[
H(z) = \frac{1}{1 - p_1 z^{-1}} + \frac{1}{1 - p_2 z^{-1}}
\]
Parallel Combination

Parallel combination of two systems looks like this:

\[ H(z) = \frac{1}{1 - p_1 z^{-1}} + \frac{1}{1 - p_2 z^{-1}} \]

\[ = \frac{1 - p_2 z^{-1}}{(1 - p_1 z^{-1})(1 - p_2 z^{-1})} + \frac{1 - p_1 z^{-1}}{(1 - p_1 z^{-1})(1 - p_2 z^{-1})} \]

\[ = \frac{2 - (p_1 + p_2) z^{-1}}{1 - (p_1 + p_2) z^{-1} + p_1 p_2 z^{-2}} \]
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Inverse Z transform

Suppose you know \( H(z) \), and you want to find \( h[n] \). How can you do that?
How to find the inverse Z transform

Any IIR filter $H(z)$ can be written as...

- a **sum** of **exponential** terms, each with this form:
  
  $$G_\ell(z) = \frac{1}{1 - az^{-1}} \quad \leftrightarrow \quad g_\ell[n] = a^n u[n],$$

- each possibly **multiplied** by a **delay** term, like this one:
  
  $$D_k(z) = b_k z^{-k} \quad \leftrightarrow \quad d_k[n] = b_k \delta[n - k].$$
Step #1: The Products

Consider one that you already know:

\[ H(z) = \frac{1 + bz^{-1}}{1 - az^{-1}} = \left( \frac{1}{1 - az^{-1}} \right) + bz^{-1} \left( \frac{1}{1 - az^{-1}} \right) \]

and therefore

\[ h[n] = (a^n u[n]) + b (a^{n-1} u[n - 1]) \]
Step #1: The Products

So here is the inverse transform of \( H(z) = \frac{1 + 0.5z^{-1}}{1 - 0.85z^{-1}} \):

\[
(0.85)^n u[n]
\]

\[
0.5(0.85)^{n-1} u[n - 1]
\]

\[
(0.85)^n u[n] + 0.5(0.85)^{n-1} u[n - 1]
\]
Step #1: The Products

In general, if

\[ G(z) = \frac{1}{A(z)} \]

for any polynomial \( A(z) \), and

\[ H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{A(z)} \]

then

\[ h[n] = b_0 g[n] + b_1 g[n - 1] + \cdots + b_M g[n - M] \]
Step #2: The Sum

Now we need to figure out the inverse transform of

\[ G(z) = \frac{1}{A(z)} \]
Step #2: The Sum

The method is this:

1. Factor $A(z)$:

\[ G(z) = \frac{1}{\prod_{\ell=1}^{N} (1 - p_{\ell}z^{-1})} \]

2. Assume that $G(z)$ is the result of a parallel system combination:

\[ G(z) = \frac{C_1}{1 - p_1z^{-1}} + \frac{C_2}{1 - p_2z^{-1}} + \cdots \]

3. Find the constants, $C_\ell$, that make the equation true.
Example

Step # 1: Factor it:

\[
\frac{1}{1 - 1.2z^{-1} + 0.72z^{-2}} = \frac{1}{(1 - (0.6 + j0.6)z^{-1})(1 - (0.6 - j0.6)z^{-1})}
\]

Step #2: Express it as a sum:

\[
\frac{1}{1 - 1.2z^{-1} + 0.72z^{-2}} = \frac{C_1}{1 - (0.6 + j0.6)z^{-1}} + \frac{C_2}{1 - (0.6 - j0.6)z^{-1}}
\]

Step #3: Find the constants. The algebra is annoying, but it turns out that:

\[
C_1 = \frac{1}{2} - j\frac{1}{2}, \quad C_2 = \frac{1}{2} + j\frac{1}{2}
\]
Example: All Done!

The system function is:

\[
G(z) = \frac{1}{1 - 1.2z^{-1} + 0.72z^{-2}} = \frac{0.5 - 0.5j}{1 - (0.6 + j0.6)z^{-1}} + \frac{0.5 + 0.5j}{1 - (0.6 - j0.6)z^{-1}}
\]

and therefore the impulse response is:

\[
g[n] = (0.5 - 0.5j)(0.6 + 0.6j)^n u[n] + (0.5 + 0.5j)(0.6 - j0.6)^n u[n]
\]

\[
= \left(0.5\sqrt{2}e^{-j\frac{\pi}{4}}\left(0.6\sqrt{2}e^{j\frac{\pi}{4}}\right)^n + 0.5\sqrt{2}e^{j\frac{\pi}{4}}\left(0.6\sqrt{2}e^{-j\frac{\pi}{4}}\right)^n\right) u[n]
\]

\[
= \sqrt{2}(0.6\sqrt{2})^n \cos\left(\frac{\pi}{4}(n - 1)\right) u[n]
\]
$g_1[n] = (0.5 - 0.5j)(0.6 + 0.6j)^nu[n]$ (imaginary part dashed)

$g_2[n] = (0.5 + 0.5j)(0.6 - 0.6j)^nu[n]$ (imaginary part dashed)

$g_1[n] + g_2[n] = (0.5\sqrt{2})(0.6\sqrt{2})^nu[n]cos(\pi(n - 1)/4)u[n]$
How to find the inverse Z transform

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A block diagram shows the delays, additions, and multiplications necessary to compute output from input.

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$$H(z) = H_2(z)H_1(z)$$

**Parallel combination**: passing the signal through two systems in parallel, then adding the outputs, is like adding the system functions:

$$H(z) = H_1(z) + H_2(z)$$
Summary: Inverse Z Transform

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Next time:

- How to design second-order notch filters, to get rid of 60Hz line noise, and...
- more about the frequency response and impulse response of second-order filters.