Lecture 1: Review of Calculus and Complex Numbers

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ECE 401: Signal and Image Analysis, Fall 2020
1. Outline of today's lecture

2. Review: How to integrate an exponential

3. Review: Summing a geometric series

4. Review: Complex numbers

5. Summary
Outline

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Outline of today’s lecture

1. Syllabus
2. Homework 1
3. Textbook
4. Review: Integration, Summation, and Complex numbers
Outline

1 Outline of today's lecture

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5 Summary
Integration = Computing the area under a curve

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https://commons.wikimedia.org/wiki/File:Integral_as_region_under_curve.svg
Real-world **signals** are functions of continuous time, or space, or both. For example, sound is air pressure as a function of time, \( p(t) \).

The **energy** necessary to produce a signal depends on its long-term integral:

\[
E \propto \int_{-\infty}^{\infty} p^2(t)dt
\]

The **information** in a signal is encoded at different frequencies \((f)\), which we can get using something called a **Fourier transform**. For continuous-time signals, a Fourier transform is an integral:

\[
P(f) = \int_{-\infty}^{\infty} p(t)e^{-j2\pi ft} dt
\]
Indefinite vs. definite integrals

- An **indefinite integral** (a.k.a. antiderivative) is the opposite of a derivative:
  \[ F(x) = \int f(x) \text{ means that } f(x) = \frac{dF}{dx} \]

- A **definite integral** is the area under the curve. We can write it as:
  \[ \int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a) \]
Indefinite integrals worth knowing

- Integral of a polynomial:
  \[ \int x^n \, dx = \frac{1}{n+1} x^{n+1} \]

- Integral of an exponential:
  \[ \int e^x \, dx = e^x \]
Methods for turning one integral into another

- Variable substitution: Suppose \( f(x) = g(u) \) where \( u \) is some function of \( x \). There is no other \( x \) anywhere inside \( f(x) \). Then:
  \[
  \int g(u) \, dx = \int \frac{1}{du/dx} g(u) \, du
  \]

- Integration by parts:
  \[
  \int u \, dv = uv - \int v \, du
  \]
Example: How to integrate an exponential

What is \( \int_{0.4}^{1.6} e^{j((x+y)t+\theta)} \, dt \)?

1. Pull out the constants:

\[
\int_{0.4}^{1.6} e^{j((x+y)t+\theta)} \, dt = e^{j\theta} \int_{0.4}^{1.6} e^{j(x+y)t} \, dt
\]

2. Prepare for variable substitution:

\[
u = j(x+y)t \quad \text{means that} \quad \frac{du}{dt} = j(x+y)
\]

\[t \in [0.4, 1.6] \quad \text{means that} \quad u \in [j(x+y)0.4, j(x+y)1.6]
\]

3. Variable substitution:

\[
\int_{0.4}^{1.6} e^{j(x+y)t} \, dt = \int_{j(x+y)0.4}^{j(x+y)1.6} \frac{1}{j(x+y)} e^{u} \, du
\]
Example: How to integrate an exponential

4. Pull out the constants again:

\[
\int_{j(x+y)0.4}^{j(x+y)1.6} \frac{1}{j(x + y)} e^u \, du = \frac{1}{j(x + y)} \int_{j(x+y)0.4}^{j(x+y)1.6} e^u \, du
\]

5. Integrate:

\[
\int_{j(x+y)0.4}^{j(x+y)1.6} e^u \, du = [e^u]_{j(x+y)0.4}^{j(x+y)1.6}
\]

6. Solve:

\[
\int_{0.4}^{1.6} e^{i((x+y)t+\theta)} \, dt = \frac{e^{i\theta}}{j(x + y)} \left( e^{j(x+y)1.6} - e^{j(x+y)0.4} \right)
\]
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Summation is a computer-friendly version of integration.
Why does signal processing use sums?

- On computers, a signal is a sequence of numbers $x[n]$, regularly spaced samples of some real-world signal.
- The **energy** necessary to produce a signal depends on its long-term summation

$$E \propto \sum_{-\infty}^{\infty} x^2[n]$$

- The **information** in a signal is encoded at different frequencies ($f$), which we can get using something called a **Fourier transform**. For discrete-time signals, a Fourier transform is a summation

$$X(\omega) = \sum_{-\infty}^{\infty} x[n]e^{-j\omega n}$$
Sums worth knowing

- Exponential series:
  \[ \sum_{n=0}^{\infty} \frac{1}{n!} x^n = e^x \]

- Geometric series:
  \[ \sum_{n=0}^{N-1} r^n = \frac{1 - r^N}{1 - r} \]
Example: How to sum a geometric series

Problem: What is $\sum_{n=-7}^{7} e^{-j\omega n}$?

1. Prepare for variable substitution #1:

   $m = n + 7$ means that $n = m - 7$

   $n \in [-7, 7]$ means that $m \in [0, 14]$

2. Variable substitution #1:

   $$\sum_{n=-7}^{7} e^{-j\omega n} = \sum_{m=0}^{14} e^{-j\omega(m-7)}$$

3. Pull out the constants:

   $$\sum_{m=0}^{14} e^{-j\omega(m-7)} = e^{7j\omega} \sum_{m=0}^{14} e^{-j\omega m}$$
Example: How to sum a geometric series

4. Prepare for variable substitution #2:

\[ r = e^{-j\omega} \] means that \[ e^{-j\omega m} = r^m \]

5. Variable substitution #2:

\[ e^{7j\omega} \sum_{m=0}^{14} e^{-j\omega m} = e^{7j\omega} \sum_{m=0}^{14} r^m \]

6. Sum:

\[ \sum_{m=0}^{14} r^m = \frac{1 - r^{15}}{1 - r} \]

7. Solve:

\[ \sum_{n=-7}^{7} e^{-j\omega n} = e^{7j\omega} \left( \frac{1 - e^{-j15\omega}}{1 - e^{-j\omega}} \right) \]
Outline of today’s lecture

Review: How to integrate an exponential

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Review: Complex numbers

Summary
Complex numbers

- **Head**
  - $z = x + jy$
- **Tail**
  - $0$
- **Real Axis**
- **Imaginary Axis**
- **Length**
  - $r$
- **Direction**
  - $\theta$

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Why does signal processing use complex numbers?

- The Fourier transform was originally defined in terms of cosines and sines:

\[ X(\omega, \theta) = \int_{-\infty}^{\infty} x(t) \cos(\omega t + \theta) dt \]

- ...but exponentials are easier to integrate than cosines, and a lot easier to sum.
- ...so we take advantage of Euler’s equation, to turn all of the cosines and sines into exponentials:

\[ e^{j\omega t} = \cos(\omega t) + j \sin(\omega t) \]
Rectangular and polar coordinates

\[ z = x + jy = me^{j\theta} \]

- Converting rectangular to polar coordinates:
  \[ m = \sqrt{x^2 + y^2}, \quad \theta = \begin{cases} 
  \text{atan} \left( \frac{y}{x} \right) & x > 0 \\
  \pm \frac{\pi}{2} & x = 0 \\
  \text{atan} \left( \frac{y}{x} \right) \pm \pi & x < 0 
\end{cases} \]

- Converting polar to rectangular:
  \[ x = m \cos \theta, \quad y = m \sin \theta \]
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1. Integration:

\[ \int e^x \, dx = e^x, \quad \int g(u) \, dx = \int \frac{1}{du/dx} g(u) \, du \]

2. Summation

\[ \sum_{n=0}^{N-1} r^n = \frac{1 - r^N}{1 - r} \]

3. Complex numbers:

\[ m = \sqrt{x^2 + y^2}, \quad \theta = \begin{cases} \arctan \left( \frac{y}{x} \right) & x > 0 \\ \pm \frac{\pi}{2} & x = 0 \\ \arctan \left( \frac{y}{x} \right) \pm \pi & x < 0 \end{cases} \]