ECE 361: Lecture 9: Energy-Efficient Communication – Part III

In Lectures 7 and 8 we considered M-ary orthogonal communication schemes which enjoy the property of being energy-efficient. In this Lecture, we consider two variations of M-ary orthogonal signaling.

- An M-ary *simplex* or *transorthogonal* signal set is obtained by translating an M-ary orthogonal set so that the translated signal set has average value 0. In effect, this eliminates the *common signal*, which, as we have seen previously, merely wastes energy without conveying any information. The M-ary simplex signal set achieves the same error probability as an M-ary orthogonal signal set for *slightly smaller* value of signal energy $\mathcal{E}_s$ or bit energy $\mathcal{E}_b$.
- An M-ary *biorthogonal* signal set consists of an $(M/2)$-ary orthogonal signal set together with the complements (negatives) of the $M/2$ signals. This allows for sending $\log_2 M$ data bits in the time that the $(M/2)$-ary orthogonal signal set sends $\log_2 M/2$ data bits so that the rate in bits per channel use is slightly larger, and $\mathcal{E}_b$ is slightly smaller. For a given signal energy $\mathcal{E}_s$, the M-ary biorthogonal signal set has slightly smaller error probability than an M-ary orthogonal signal set.

While both the transorthogonal and biorthogonal signal sets improve in small ways on the performance of M-ary orthogonal signaling, these improvements do not have any effect on the asymptotic performance. In particular, the result noted in Lecture 8 that $P(E)$ can be made arbitrarily small by increasing M if and only if $\mathcal{E}_b > 2\sigma^2 \ln 2$ still applies.

9.1. M-ary Simplex or Transorthogonal Signaling

9.1.1. The Translated Signal Set

Let $\underline{s}_0, \underline{s}_1, \ldots, \underline{s}_{M-1}$ denote M orthogonal signals each of energy $\mathcal{E}_s$. Thus, we have

$$\| \underline{s}_i \| = \sqrt{\mathcal{E}_s} \text{ for all } i, \text{ and } \langle \underline{s}_i, \underline{s}_j \rangle = 0, \quad \| \underline{s}_i - \underline{s}_j \| = \sqrt{2\mathcal{E}_s} \text{ for } i \neq j.$$

Let $\underline{\bar{s}} = M^{-1} \sum_i \underline{s}_i$ denote the average signal or *common signal* which has energy $\| \underline{\bar{s}} \|^2 = M^{-1}\mathcal{E}_s$. The simplex or transorthogonal signal set is defined as $\underline{\hat{s}}_i = \underline{s}_i - \underline{\bar{s}}$ for $0 \leq i \leq M-1$. Note that while the original signals are *orthogonal* and can be thought of as the basis vectors of an M-dimensional space, the translated signals $\underline{\hat{s}}_i$ satisfy $\sum_i \underline{\hat{s}}_i = 0$ and thus cannot be a basis: they are not linearly independent. In fact the M translated signals occupy an $(M-1)$-dimensional space. The translated signals are the vertices of what mathematicians call a *regular simplex*, and hence are called a *simplex signal set.*[1] Since we have removed the common signal from the signals, it should not be too surprising that the translated signals have slightly less energy. We have that

$$\| \underline{\hat{s}}_i \|^2 = \| \underline{s}_i - \underline{\bar{s}} \|^2 = \| \underline{s}_i \|^2 + \| \underline{\bar{s}} \|^2 - 2\langle \underline{s}_i, \underline{\bar{s}} \rangle = \mathcal{E}_s + \frac{1}{M}\mathcal{E}_s - \frac{2}{M}\mathcal{E}_s = \mathcal{E}_s \left(1 - \frac{1}{M}\right) < \mathcal{E}_s.$$

The translated signals are not orthogonal either. For $i \neq j$,

$$\langle \underline{\hat{s}}_i, \underline{\hat{s}}_j \rangle = \langle \underline{s}_i - \underline{\bar{s}}, \underline{s}_j - \underline{\bar{s}} \rangle = \langle \underline{s}_i, \underline{s}_j \rangle - \langle \underline{s}_i, \underline{\bar{s}} \rangle - \langle \underline{s}_j, \underline{\bar{s}} \rangle + \| \underline{\bar{s}} \|^2 = -\frac{\mathcal{E}_s}{M},$$

but the *distances* between the signals are the same:

$$\| \underline{\hat{s}}_i - \underline{\hat{s}}_i \| = \| \underline{s}_i - \underline{\bar{s}} - (\underline{s}_j - \underline{\bar{s}}) \| = \sqrt{2\mathcal{E}_s} \text{ for } i \neq j.$$

[1] We will not be needing any more than the name "simplex" in this course, but the following examples are helpful in understanding the concept. For $M = 3$, the tips of the three orthogonal vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ in three-dimensional space are the vertices of an equilateral triangle, and this triangle is the regular simplex. Note that the vertices lie in a plane, *viz.* a 2-dimensional space. For $M = 4$, the regular simplex is a regular tetrahedron, and the 4 vertices are in a 3-dimensional space.

Consequently, the error probabilities achieved are exactly the same as with M-ary orthogonal signaling but the energy required is slightly less by a factor of $(M-1)/M$. As M increases, the savings become smaller and smaller, and, as noted before, the asymptotic performance is the same as with M-ary orthogonal signaling.

9.1.2. Two Special Cases

1. PPM signaling with an average power constraint but no peak power constraint

In Lecture 7, we considered the PPM signal set

$$\begin{array}{lcllcl} \underline{s}_0 & = & [\sqrt{\mathcal{E}_s}, & 0, & \ldots & 0\ ] \\ \underline{s}_1 & = & [\ 0, & \sqrt{\mathcal{E}_s}, & \ldots & 0\ ] \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \underline{s}_{M-1} & = & [\ 0, & 0, & \ldots & \sqrt{\mathcal{E}_s}\,] \end{array}$$

Obviously, the average signal is $\bar{\underline{s}} = M^{-1}[\sqrt{\mathcal{E}_s}, \sqrt{\mathcal{E}_s}, \ldots, \sqrt{\mathcal{E}_s}]$ so that

$$\begin{array}{lcllcl} \hat{\underline{s}}_0 & = & [\ \frac{M-1}{M}\sqrt{\mathcal{E}_s}, & -\frac{1}{M}\sqrt{\mathcal{E}_s}, & \ldots & -\frac{1}{M}\sqrt{\mathcal{E}_s}\ ] \\ \hat{\underline{s}}_1 & = & [\ -\frac{1}{M}\sqrt{\mathcal{E}_s}, & \frac{M-1}{M}\sqrt{\mathcal{E}_s}, & \ldots & -\frac{1}{M}\sqrt{\mathcal{E}_s}\ ] \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \hat{\underline{s}}_{M-1} & = & [\ -\frac{1}{M}\sqrt{\mathcal{E}_s}, & -\frac{1}{M}\sqrt{\mathcal{E}_s}, & \ldots & \frac{M-1}{M}\sqrt{\mathcal{E}_s}\,] \end{array}$$

is a simplex signal set. The amplitude of the large pulse is reduced by a factor of $(M-1)/M$, but small negative pulses are transmitted at other times. Note that the large pulse has energy $((M-1)/M)^2\mathcal{E}_s$ while the small pulses together contribute $(M-1)\cdot(1/M)^2\mathcal{E}_s$ for a total energy of $((M-1)/M)\mathcal{E}_s$.

The receiver in an M-ary transorthogonal communication system using PPM is the same as the receiver in an M-ary orthogonal communication system using PPM. The receiver has available to it M random variables $\mathbb{Y}_0$, $\mathbb{Y}_1$, …, $\mathbb{Y}_{M-1}$. As before, if the largest of the observations $\mathbb{Y}_0$, $\mathbb{Y}_1$, …, $\mathbb{Y}_{M-1}$ is $\mathbb{Y}_j$, then the receiver decides that s_j was the transmitted signal. However, now there is a minor twist in the detailed analysis which can be eliminated easily. Given that signal s_i was transmitted, the $\mathbb{Y}$'s are conditionally independent Gaussian random variables with common variance σ^2 with means $\mathsf{E}[\mathbb{Y}_i] = \frac{M-1}{M}\sqrt{\mathcal{E}_s}$, $\mathsf{E}[\mathbb{Y}_j] = -\frac{1}{M}\sqrt{\mathcal{E}_s}$ for $j \neq i$. As in Lecture 8, let C denote the event that the receiver decision is correct. Since we are assuming that s_i is the transmitted signal, we have that

$$P(C \mid s_i \text{ transmitted}) = P\big\{\mathbb{Y}_i > \max\{\mathbb{Y}_0, \mathbb{Y}_1, \ldots, \mathbb{Y}_{i-1}, \mathbb{Y}_{i+1}, \ldots, \mathbb{Y}_{M-1}\}\big\}$$

To calculate this probability, let us suppose that $\mathbb{Y}_i = \alpha$. Then,

$$\begin{aligned} P(C \mid s_i \text{ transmitted}, \mathbb{Y}_i = \alpha) &= P\{\mathbb{Y}_0 < \alpha, \mathbb{Y}_1 < \alpha, \ldots, \mathbb{Y}_{i-1} < \alpha, \mathbb{Y}_{i+1} < \alpha, \ldots, \mathbb{Y}_{M-1} < \alpha\} \\ &= P\{\mathbb{Y}_0 < \alpha\}P\{\mathbb{Y}_1 < \alpha\}\cdots P\{\mathbb{Y}_{i-1} < \alpha\}P\{\mathbb{Y}_{i+1} < \alpha\}\cdots P\{\mathbb{Y}_{M-1} < \alpha\} \\ &= \big[\Phi((\alpha + M^{-1}\sqrt{\mathcal{E}_s}/\sigma)\big]^{M-1}. \end{aligned}$$

The law of total probability tells us that we can remove the conditioning on the value of $\mathbb{Y}_i$ by multiplying by the pdf $\sigma^{-1}\phi((\alpha - \frac{M-1}{M}\sqrt{\mathcal{E}_s})/\sigma)$ of $\mathbb{Y}_i$ and integrating with respect to α. Thus we have (after making a change of variable $x = (\alpha + \frac{1}{M}\sqrt{\mathcal{E}_s})/\sigma$) that

$$P(C) = \int_{-\infty}^{\infty} \left[\Phi\left(\frac{\alpha + \frac{1}{M}\sqrt{\mathcal{E}_s}}{\sigma}\right)\right]^{M-1} \frac{1}{\sigma}\phi\left(\frac{\alpha - \frac{M-1}{M}\sqrt{\mathcal{E}_s}}{\sigma}\right) d\alpha = \int_{-\infty}^{\infty} [\Phi(x)]^{M-1}\,\phi(x-\mu)\,dx \qquad (9.1)$$

where $\mu = \sqrt{\mathcal{E}_s/\sigma^2}$. This is the same as (8.1) in Lecture 8, showing that the transorthogonal signaling scheme has the same error probability as the M-ary orthogonal signaling scheme, but using energy smaller by a factor of $(M-1)/M$. As noted previously, the advantage gained is small when M is large.

2. Hadamard coding with a peak power constraint

Let $M = 2^B$ and consider the M-ary orthogonal signaling set

$$\underline{s}_{i,H} = \sqrt{\mathcal{E}_s/M}\Big[\, h_{i,0}, h_{i,1}, \ldots, h_{i,M-1} \Big], \quad 0 \le i \le M-1$$

of energy $\mathcal{E}_s$. Assume that the Hadamard matrix has been obtained via the Sylvester construction so that $h_{i,0} = +1$ for all i. Also, all other columns have equal numbers of $+1$ and -1 entries. Consequently, the average signal is

$$\underline{\bar{s}} = \sqrt{\mathcal{E}_s/M}\Big[\, 1, 0, \ldots, 0 \Big],$$

and the translated signals are

$$\underline{\hat{s}}_{i,H} = \sqrt{\mathcal{E}_s/M}\Big[\, 0, h_{i,1}, \ldots, h_{i,M-1} \Big], \quad 0 \le i \le M-1.$$

Note that the translated signals have energy $\frac{M-1}{M}\mathcal{E}_s$. In practice, the zero-th symbol is not transmitted at all, so that a small advantage in rate is possible since we can transmit B bits with only $2^B - 1$ channel uses instead of 2^B channel uses.

Suppose that the zero-th symbol has *not* been deleted from each of the translated signals. Then, the receiver in an M-ary transorthogonal communication system using Hadamard coding has M random variables to which it applies the inverse Hadamard transform as described in Section 2 of Lecture 7, and then the receiver finds the largest output value of the inverse Hadamard tranform to decide which signal was sent. The analysis presented in Lecture 8 thus applies and the performance is as described there. But notice that the input $\mathbb{Y}_0$ to the inverse Hadamard transform is purely a noise variable since no signal was transmitted during that channel use. Furthermore, since the first row of the Hadamard matrix is all $+1$'s, this noise random variable $\mathbb{Y}_0$ simply adds to all the inverse transform values and *does not affect which of the inverse transform values is the largest.* Thus, the receiver can simply ignore $\mathbb{Y}_0$ or set it to 0 without affecting anything. And if the receiver is going to ignore $\mathbb{Y}_0$, it does not really need the zero-th channel use in order to work. Thus, as noted above, the zero-th symbol in $\hat{s}_i$ is not transmitted at all, and the receiver has available to it just $M-1$ random variables $\mathbb{Y}_i$, $1 \le i \le M-1$. The receiver multiplies $[\mathbb{Y}_1, \mathbb{Y}_2, \ldots, \mathbb{Y}_{M-1}]$ by the $(M-1) \times M$ matrix obtained by deleting the first row of the Hadamard matrix, and chooses the largest of the inverse transform values, etc. The error probability is the same as with M-ary orthogonal signaling but is achieved with slightly smaller energy.

9.2. M-ary Biorthogonal Signaling

Throughout this section, we assume that M is an even integer. An M-ary biorthogonal signal set consists of $M/2$ orthogonal signals together with their negatives. It is convenient to use the notation

$$s_{+0},\ s_{+1},\ \ldots,\ s_{+(M/2-1)},\ s_{-0},\ s_{-1},\ \ldots,\ s_{-(M/2-1)}$$

to denote the signals where the first $M/2$ signals (with positive signs in the subscripts) are orthogonal signals and $s_{-i} = -s_{+i}$ for $0 \le i \le M/2 - 1$. We refer to s_{+i} and s_{-i} as *complementary* signals or complements of each other. Notice that *any* of the M signals is complementary to exactly one of the other $M-1$ signals and orthogonal to the rest of them. Consequently, with $\mathcal{E}_s$ denoting the common energy of the signals, we have

$$\| s_i - s_j \| = \begin{cases} 0, & \text{if } i = j, \\ 2\sqrt{\mathcal{E}_s}, & \text{if } i = -j, \\ \sqrt{\mathcal{E}_s}, & \text{otherwise,} \end{cases}$$

so that each signal has $M-2$ nearest neighbors and one distant neighbor (who happens to be a close relative!) With PPM signaling, we can take s_{+i} to be a single positive pulse during the i-th of $M/2$ channel uses and s_{-i} to be a single negative pulse during the i-th of $M/2$ channel uses. Thus, the receiver has $M-2$ random variables $\mathbb{Y}_0$, $\mathbb{Y}_1$, ..., $\mathbb{Y}_{M/2-1}$ where, conditioned on $s_{\pm i}$ being transmitted, these are conditionally independent Gaussian random variables with variance σ^2 and means $\mathsf{E}[\mathbb{Y}_i] = \pm\sqrt{\mathcal{E}_s}$, $\mathsf{E}[\mathbb{Y}_j] = 0$ for $j \neq i$. The receiver decides which signal was transmitted by first finding the largest of $|\mathbb{Y}_0|$, $|\mathbb{Y}_1|$, ..., $|\mathbb{Y}_{M/2-1}|$. If $|\mathbb{Y}_j|$ is the largest magnitude, then the receiver decides that s_{+j} or s_{-j} was transmitted according as $\mathbb{Y}_j > 0$ or $\mathbb{Y}_j < 0$.

Suppose that s_{+i} is the transmitted signal. Then, conditioned on $\mathbb{Y}_i = x\sigma$ where $\alpha > 0$, the probability of a correct decision is the probability that all the other $\mathbb{Y}_j$ have values in $(-x\sigma, +x\sigma)$, which is $[\Phi(x) - \Phi(-x)]^{M/2-1}$. On the other hand, if $\mathbb{Y}_i = x\sigma$ where $\alpha < 0$, then the receiver will not make a correct decision at all. *At best*, it will decide that s_{-i} was transmitted, which is still wrong. From this, following the procedure in Lecture 8, we get that

$$P(C) = \int_0^\infty [\Phi(x) - \Phi(-x)]^{M/2-1} \phi(x-\mu)\,dx = \int_0^\infty [1 - 2Q(x)]^{M/2-1} \phi(x-\mu)\,dx$$

where $\mu = \sqrt{\mathcal{E}_s/\sigma^2}$. Note the lower limit of 0 in the integral, and the different integrand as compared to (8.1). This result was of course obtained conditioned on s_{+i} being the transmitted signal, but the analysis is similar when we assume that s_{-i} is the transmitted signal, and a simple change of variable converts the integral from $-\infty$ to 0 into the one shown above.

As in Lecture 8, $P(E)$ can be obtained by integrating by parts, and we get

$$P(E) = \int_0^\infty (M-2)\,[1 - 2Q(x)]^{M/2-2}\, \Phi(x-\mu)\phi(x)\,dx.$$

The union bound on $P(E)$ has a slightly different form than for M-ary orthogonal signaling. We have

$$P(E) < (M-2)Q(\mu/\sqrt{2}) + Q(\mu)$$

which is slightly smaller than the union bound $(M-1)Q(\mu/\sqrt{2})$ for M-ary orthogonal signaling. The difference is essentially due to the fact that in M-ary orthogonal signaling, each signal has $M-1$ nearest neighbors while in M-ary biorthogonal signaling, each signal has $M-2$ nearest neighbors and one distant neighbor. In fact, it is not just the bound but the error probability that is slightly smaller. Conversely, to achieve a given error probability, biorthogonal signals require slightly less energy than orthogonal signals.

Of the $M-1$ different errors that the receiver can make, mistaking s_{+i} for s_{-i} or vice versa is the most unlikely. Thus, if one of the bits is more important than the others, and needs or deserves extra protection, then it should be encoded into this transition, that is, the bit patterns assigned to s_{+i} and s_{-i} should differ in just this most important bit. In the more usual case when all the bits are equally important, the bit patterns assigned to s_{+i} and s_{-i} are *complements* of each other so that the worst possible error (making a mistake in each and every bit) has the least probability.

9.3. Summary

Both simplex signal sets and biorthogonal signal sets require slightly smaller signal energy than orthogonal signal sets to achieve the same error probability. Conversely, for a given signal energy $\mathcal{E}_s$ or energy per bit $\mathcal{E}_b$, simplex signal sets and biorthogonal signal sets achieve slightly smaller error probability. Both the simplex signal set and the biorthogonal signal set have average signal equal to 0 which can be important in some applications.