ECE 361: Problem Set *: Problems and Solutions
Maximum A Posteriori estimation practice

Released: Tuesday, January 19
Due: Thursday, January 28 (in class)
Reading: 361 Course Notes Lectures 1-3, and S1.

1. [Weather Forecast]
   by Vincent Ballet

Air pressure of the preceding day is an important indicator of the weather of the following day. Our task is to predict whether it is going to rain ($H = 1$) or not ($H = 0$) tomorrow. The two hypotheses are equally likely. The air pressure of today $Y$, normalized to range in $[0,1]$, follows the following conditional probability density distribution.

$$p_{Y\mid H}(y\mid H = 0) = \alpha - \frac{\alpha}{5} y, y \in [0,1]$$
$$p_{Y\mid H}(y\mid H = 1) = \beta - \frac{\beta}{5} y, y \in [0,1]$$

(a) Find $\alpha$ and $\beta$
   Solution:

   $$\alpha = \frac{9}{10}$$
   $$\alpha = \frac{4}{5}$$

(b) Find $P_{H\mid Y}(0\mid y)$ and $P_{H\mid Y}(1\mid y)$
   Solution:

   $$P_{H\mid Y}(0\mid y) = \frac{5(5 - y)}{4x + 43}$$
   $$P_{H\mid Y}(1\mid y) = \frac{9(y + 2)}{4x + 43}$$

(c) Define the decision rule for $H$
   Solution: $H = 0$ if $y \leq 0.5$, and $H = 1$ otherwise

(d) Compute the probability of error. In this case, is the air pressure a good predictor of weather?
   Solution: 0.46. This is a pretty bad predictor.

2. [Motion Detector]
   by Gregory Wajda

Two detector are used to detect motion in a room. The first sensor outputs $Y$, and the second outputs $Z$. Both $Y$ and $Z$ are random variables that take 3 possible values: $\{0, 1, 2\}$. The higher the values are, the more likely there is motion detected. More specifically, let $H = 1$ denote there’s some motion, and $H = 0$ denote there is not. The prior probability of a motion taking place is

$$P(H = 0) = 0.8, P(H = 1) = 0.2$$

The conditional probability $P_{Y\mid H}$ and $P_{Z\mid H}$ are given in the following table

The two sensors work independently, conditional on the motion status in the room. Compute the joint posterior probability $P_{H\mid Y,Z}$ and indicate the MAP decision rule.
The posterior distribution is given by

Through inspection, the MAP rule is

\[
\hat{H} = \begin{cases} 
1 & \text{if } Y \geq 1 \text{ and } Z = 2 \\
0 & \text{otherwise}
\end{cases}
\]

3. **[Ternary Communication Channel]**
   *by Alican Akman*

Consider a communication system where the received signal \( Y \) is

\[ Y = X + W \]

\( X \) can take \( \{-2, 0, 2\} \), and \( W \) is zero mean Gaussian noise with variance 1.

(a) Find the MAP decision rule when \( P(X = -2) = P(X = 2) = 1/4, P(X = 2) = 1/2 \).

**Solution:** The two thresholds are

\[ \pm \ln 2 + 2 \]

(b) Let \( P(X = -2) = P(X = 2) = 0.5 - 0.5\theta, P(X = 2) = \theta \). Find the maximum value of \( \theta \) such that the MAP decision rule would never choose \( X = 0 \) (Hence the channel essentially collapses to a binary channel).

**Solution:**

\[ \theta_{\text{max}} = \frac{1}{1 + 2e^{-2}} \]

4. **[Channel with Assymmetric Noise]**
   *by Michael Silkaitis*

Consider a binary communication system where the received signal \( Y \) is:

\[ Y = X + W \]

where \( X \) can take values \( a_0 = -0.5 \) or \( a_1 = 0.5 \). However, the channel’s noise attenuates higher voltages, and dependent upon the signal voltage. The resulting conditional density functions are

\[
p_{Y|X}(y|x = a_0) = \begin{cases} 
\frac{3}{4} - x & -\frac{1}{2} \leq x \leq 0 \\
0 & \text{otherwise}
\end{cases}
\]

and

\[
p_{Y|X}(y|x = a_1) = \begin{cases} 
\frac{3}{4} + \frac{9}{20}x & 0 \leq x \leq \frac{5}{3} \\
0 & \text{otherwise}
\end{cases}
\]

(a) Assuming both \( a_0 \) and \( a_1 \) are equally likely, find the MAP rule for this channel.

**Solution:**

\[ \hat{x} = \begin{cases} 
a_1 & y \geq -0.5 \\
a_0 & y < -0.5
\end{cases} \]
(b) Compute the probability of error $p_e$.

**Solution:**

$$p_e = \frac{11}{64}$$

(c) Suppose now the prior for $a_0$ is $\theta, \theta \in [0, 1]$. Find the set of $\theta$ under which no error would be made by the MAP rule when $a_0$ is sent. Also, find the set of $\theta$ under which no error would be made by the MAP rule when $a_1$ is sent.

**Solution:**

$$\theta \in \left[\frac{3}{5}, 1\right], \theta \in \left[0, \frac{3}{7}\right]$$

5. **[AWGN Channel in Thunderstorm]**

*by Dennis Ryu*

Suppose we have a vulnerable AWGN channel that sends a signal once every second. In the meantime, however, a harsh storm approaches, causing frequent lightening, which induces huge electric shocks in the channel. Suppose without the lightening ($H = 0$), the received signal $X$ follows a normal distribution $\mathcal{N}(0, 1)$. When there is lightening ($H = 1$), the huge electric shock alters the distribution of $X$ to Laplacian distribution:

$$p_{X|H}(x|H = 1) = \frac{1}{20} \exp \left(-\frac{|x|}{10}\right)$$

Suppose the lightening happens roughly once every minute. That is $P(H = 1) = 1/60$. Derive the MAP rule to determine if there is a lightening based on the received signal $X$.

**Solution:** Not yet available.