ECE 361: Problem Set 4: Problems and Solutions
Capacity and Sampling

Released: Tuesday, February 16
Due: Thursday, February 25 (in class)
Reading: 361 Course Notes Lectures 11-13.

1. [BEC with Feedback]
Suppose we are communicating over a BEC and there is feedback, as present in internet communication. In particular, assume the transmitter is told the output of the channel, \( y[1], \ldots, y[i-1] \) at time \( i \), so that it knows which transmitted symbols were erased. Suppose the probability of erasure is \( p \), and we use the following transmission scheme:

- Transmit \( n \) symbols,
- We know approximately \( pn \) were erased, so we retransmit them.
- We know approximately \( p \) fraction of these were erased, so we retransmit them.
- We continue following the third step until all symbols have been received unerased.

(a) In expectation, how many total transmissions are required until all symbols are received without erasure?

Solution:

\[
n + pn + p^2n + \cdots = \sum_{j=0}^{\infty} j = 0^\infty p^j n \\
= n \sum_{j=0}^{\infty} j = 0^\infty p^j \\
= \frac{n}{1-p} \\
= \frac{n}{1-p}
\]

(b) What is the (expected) communication rate of this scheme?

Solution: To get \( n \) bits, we used \( n/(1-p) \) channel usages so the rate is

\[
R = \frac{n}{n/(1-p)} = 1 - p
\]

This is the same as the capacity without feedback we saw in lecture. In fact Shannon (1956) showed that surprisingly, feedback does not increase the capacity of any memoryless channel.

2. [Minimum SNR for Reliable Communication]
With an average energy constraint of \( E \), the capacity of the discrete-time Gaussian channel was noted to be \( C = \frac{1}{2} \log_2 (1 + \text{SNR}) \) where \( \text{SNR} = E/\sigma^2 \). Now consider a communication system operating at rate \( R \) very close to \( C \) with very high reliability.

(a) Show that the bit energy \( E_b \) equals \( E/R \).

Solution: We transmit \( k \) bits in \( n \) channel uses using a total energy \( nE \). Hence, \( E_b = nE/k = E/R \).

(b) Show that \( E_b/\sigma^2 \approx 2\text{SNR}/\log_2 (1 + \text{SNR}) \) which is an increasing function of \( \text{SNR} \).

Solution: \( E_b/\sigma^2 = \text{SNR}/R \), but since \( R \approx C = \frac{1}{2} \log_2 (1 + \text{SNR}) \), we have \( E_b/\sigma^2 \approx \frac{2\text{SNR}}{\log_2 (1 + \text{SNR})} \).

Since \( \text{SNR} \) increases faster than \( \log_2 (1 + \text{SNR}) \), the ratio should be an increasing function of \( \text{SNR} \). This can be proved more formally by taking the derivative with respect to \( \text{SNR} \) and showing that the derivative is positive for all \( \text{SNR} > 0 \).
(c) Naturally, we want to operate with as small a value of $E_b$ as possible. Show that $\lim_{\text{SNR} \to 0} \frac{2\text{SNR}}{\log_2 (1 + \text{SNR})} = 2 \ln 2 \approx 1.4$ and so it is possible to operate very close to capacity with very high reliability provided that $E_b/\sigma^2 > 2 \ln 2$.

**Solution:** Since $\frac{2\text{SNR}}{\log_2 (1 + \text{SNR})}$ is of the form $0/0$ at $\text{SNR} = 0$, we need to use l'Hôpital’s rule which gives

$$\lim_{\text{SNR} \to 0} \frac{2\text{SNR}}{\log_2 (1 + \text{SNR})} = \lim_{\text{SNR} \to 0} \frac{2\text{SNR} \ln 2}{1/(1 + \text{SNR})} = \lim_{\text{SNR} \to 0} \frac{2\ln 2}{1/(1 + \text{SNR})} = 2 \ln 2.$$  

Thus, the limiting value of $E_b/\sigma^2$ is $2 \ln 2$. We need a bit energy to noise variance ratio larger than $2 \ln 2$ to operate with high reliability rates close to capacity. The converse statement is also true. If $E_b/\sigma^2 < 2 \ln 2$, then it is not possible to operate with high reliability at any rate.

3. **[Raised Cosine Pulse]**

The raised cosine pulse in the time domain is given by:

$$g_{rc}(t) = \text{sinc} \left( \frac{t}{T} \right) \cos \left( \frac{\pi t}{T} \right) \frac{\sin \left( \frac{\pi \beta t}{T} \right)}{1 - \frac{4\beta^2 t^2}{T^2}}$$

In this problem you will establish that its Fourier transform is given by

$$G_{rc}(f) = \begin{cases} \frac{T}{\pi} [1 + \cos \left( \frac{\pi f}{\beta} \left( |f| - \frac{1 - \beta}{2T} \right) \right)] & 0 \leq |f| \leq \frac{1 - \beta}{2T} \\ 0 & \frac{1 - \beta}{2T} \leq |f| \leq \frac{1 + \beta}{2T} \\ \frac{1}{1 - \frac{4\beta^2 t^2}{T^2}} \end{cases}$$

(a) First show that $g_{rc}(t)$ can be rewritten as:

$$g_{rc}(t) = \text{sinc} \left( \frac{t}{T} \right) b(t)$$

where

$$b(t) = \frac{\pi}{4} \left[ \text{sinc} \left( \frac{\beta t}{T} + \frac{1}{2} \right) + \text{sinc} \left( \frac{\beta t}{T} - \frac{1}{2} \right) \right]$$

**Solution:** It is easier to work backwards as follows:

$$b(t) = \frac{\pi}{4} \frac{\sin \left( \frac{\pi \beta t}{T} + \frac{\pi}{2} \right)}{\pi \beta t / T + \frac{\pi}{2}} + \frac{\pi}{4} \frac{\sin \left( \frac{\pi \beta t}{T} - \frac{\pi}{2} \right)}{\pi \beta t / T - \frac{\pi}{2}} = \cos \left( \frac{\pi \beta t}{T} \right) \frac{4\beta t}{T^2} + \frac{4\beta t}{T^2} = \frac{1}{1 - \frac{4\beta^2 t^2}{T^2}}$$

(b) Next show that the Fourier transform of $b(t)$ is given by

$$B(f) = \frac{\pi T}{2\beta} \cos \left( \frac{\pi T f}{\beta} \right) \text{rect} \left( \frac{fT}{\beta} \right)$$

**Solution:** We use the fact that $\text{sinc} \left( \frac{t}{T} \right) \leftrightarrow T \text{rect}(fT)$ to conclude that

$$\text{sinc} \left( \frac{\beta t}{T} + \frac{1}{2} \right) = \text{sinc} \left( \frac{\beta}{T} \left( t + \frac{T}{2\beta} \right) \right) \leftrightarrow \frac{T}{\beta} \text{rect} \left( \frac{fT}{\beta} \right) e^{j2\pi f \frac{T}{\beta}}$$

and

$$\text{sinc} \left( \frac{\beta t}{T} - \frac{1}{2} \right) = \text{sinc} \left( \frac{\beta}{T} \left( t - \frac{T}{2\beta} \right) \right) \leftrightarrow \frac{T}{\beta} \text{rect} \left( \frac{fT}{\beta} \right) e^{-j2\pi f \frac{T}{\beta}}$$
Therefore
\[ B(f) = \frac{\pi}{4} T \beta \text{rect} \left( \frac{fT}{\beta} \right) \left[ e^{j2\pi fT \beta} + e^{-j2\pi fT \beta} \right] = \frac{\pi T}{2\beta} \cos \left( \frac{\pi T f}{\beta} \right) \text{rect} \left( \frac{Tf}{\beta} \right) \]

(c) Now find \( G_{rc}(f) \) by convolving \( B(f) \) with \( T \text{rect}(Tf) \).

\text{Hint: } Only consider \( f \geq 0 \) in computing the convolution, arguing first that \( G_{rc}(f) \) is an even function.

\textbf{Solution:} Since \( B(f) \) and \( T \text{rect}(Tf) \) are even functions of \( f \), their convolution is also an even function. For \( 0 \leq f < 1 - \frac{\beta}{2T} \)

\[ G_{rc}(f) = \int_{-\infty}^{\infty} T \text{rect}(T\phi)B(f-\phi) d\phi = \frac{\pi T^2}{2\beta} \int_{f-\frac{\beta}{2T}}^{f+\frac{\beta}{2T}} \cos \left( \frac{\pi T(f-\phi)}{\beta} \right) d\phi \]

\[ = \frac{T}{2} \left[ \sin \left( \frac{\pi T(f-\phi)}{\beta} \right) \right]_{f-\frac{\beta}{2T}}^{f+\frac{\beta}{2T}} = \frac{T}{2} [\sin(\pi/2) - \sin(-\pi/2)] = T \]

Similarly, for \( 1 - \frac{\beta}{2T} \leq f \leq 1 + \frac{\beta}{2T} \),

\[ G_{rc}(f) = \frac{\pi T^2}{2\beta} \int_{f-\frac{\beta}{2T}}^{f+\frac{\beta}{2T}} \cos \left( \frac{\pi T(f-\phi)}{\beta} \right) d\phi = \frac{T}{2} \left[ \sin \left( \frac{\pi T(f-\phi)}{\beta} \right) \right]_{f-\frac{\beta}{2T}}^{f+\frac{\beta}{2T}} \]

\[ = \frac{T}{2} \left[ \sin \left( \frac{\pi T(f-\phi)}{\beta} \right) - \sin(-\pi/2) \right] = \frac{T}{2} \left[ 1 + \cos \left( \frac{\pi T}{\beta} \left( f - 1 - \frac{\beta}{2T} \right) \right) \right] \]

and \( G_{rc}(f) = 0 \) for \( f > 1 + \frac{\beta}{2T} \).

4. \textit{[Minimum Energy per bit and Optimality of Position Modulation]}

Consider communication on a continuous-time AWGN channel with bandwidth \( W \) and power \( P \). The energy per bit that is consumed if we transmit bits at rate \( R \) bits/s is given by

\[ E_b = \frac{P}{R} \]

For a fixed bandwidth \( W \), the minimum energy per bit is therefore obtained when we communicate at the capacity, i.e., when 

\[ R = C_{\text{AWGN}} = \frac{W}{2} \log_2 \left( 1 + \frac{2P}{N_0 W} \right) \text{ bits/s.} \]

and the corresponding energy per bit is given by

\[ E_b(W) = \frac{2P}{W \log_2 \left( 1 + \frac{2P}{N_0 W} \right)} \]

(a) Show that the minimum value of \( E_b(W) \), minimized over \( W \), is given by \( E_{b,\text{min}} = N_0 \ln 2 \).

\textbf{Solution:} The minimum value of \( E_b(W) \) over \( W \) is obtained as \( W \to \infty \), since \( C_{\text{AWGN}} \) is maximized as \( W \to \infty \). Therefore, using eq (24) of the Lecture 10 notes, we can conclude that

\[ E_{b,\text{min}} = \frac{PN_0}{P \log_2 e} = N_0 \ln 2 \]

(b) Now consider applying the position modulation scheme of Lecture 5 of the notes on the continuous time AWGN channel with bandwidth \( W \), by using the pulse shaping and sampling scheme discussed in Lecture 9 of the notes. We know from Lecture 5 that the minimum energy per bit required for reliable position modulation on the discrete-time AWGN channel with noise variance
\[ \sigma^2 \text{ is given by } 2\sigma^2 \ln 2. \text{ Now, assuming that the samples are sent at the the Nyquist rate of } W \text{ samples per second on the continuous-time AWGN channel (using the ideal sinc pulse), argue that position modulation achieves the } E_{b, \text{min}} \text{ of part (a).} \]

**Solution:** As we discussed in class, the noise power in bandwidth \( W \) is given by \( N_0W \). Therefore, the energy in the noise per sample, \( \sigma^2 \) is equal to \( \frac{N_0W}{T} \), where \( T \) is the sample period. If we sample at the Nyquist rate, then \( T = \frac{1}{W} \). Therefore \( \sigma^2 = \frac{N_0}{2} \), which means that \( 2\sigma^2 \ln 2 = N_0 \ln 2 \).

5. **[Prolate Spheroidal Wave Functions]**

This problem considers pulse-shaping.

(a) In Lecture, we played around with the tradeoff between bandwidth and time extent in the choice of raised cosine pulse shapes (see posted matlab code), but did not formalize a definition of bandwidth for time-limited signals. Define a precise notion of bandwidth (also write this down for us), and plot this bandwidth as a function of the parameter \( \beta \).

**Solution:** Depends on the definition for bandwidth chosen.

(b) In Lecture, we mentioned the prolate spheroidal wave function as a good pulse shape. The posted matlab code computes a discrete-time approximation. Try to study the bandwidth of this pulse shape in the same way as for the raised cosine. It would probably make sense to use parameter values \( W = 0.1 \) and less.

**Solution:** Depends on the definition for bandwidth chosen.

6. **[Gaussian Channel Capacity]**

Suppose signals of average power \( P \) are used over a white Gaussian noise channel with power spectral density \( N_0/2 \) Watts/Hz and with bandwidth \( W \) Hz. Then the channel capacity \( C \) is given by:

\[
C = W \log_2 \left( 1 + \frac{P}{N_0W} \right) \text{ bits/second.}
\]

(a) For fixed \( \text{SNR} = \frac{P}{N_0} \), as \( W \) increases without bound (tends to \( \infty \)), does \( C \) also increase without bound? If not, what is \( \lim_{W \to \infty} C \) and does \( C \) approach its limiting value monotonically from below, monotonically from above, or in some non-monotone fashion?

**Solution:** \( C \) approaches a limit as \( W \to \infty \) and it approaches the limit from below. We have that

\[
\lim_{W \to \infty} W \log_2 \left( 1 + \frac{P}{N_0W} \right) = \lim_{W \to \infty} \frac{W}{\ln 2} \ln \left( 1 + \frac{P}{N_0W} \right) = \lim_{W \to \infty} W \left[ \frac{P}{N_0W} - \frac{1}{2} \left( \frac{P}{N_0W} \right)^2 + \frac{1}{3} \left( \frac{P}{N_0W} \right)^3 - \cdots \right] = \frac{P}{N_0} \ln 2 = \frac{P}{N_0} \log_2 e.
\]

The key limitation on the capacity of wideband channels is the power in the signal. Put another way, \( \frac{dC}{dW} \), the derivative of \( C \) with respect to \( W \), is small when \( W \) is large, and increasing an already large bandwidth does not increase the capacity of the channel very much. Wideband channels are said to be power-limited; the capacity depends primarily on the power of the signal, not the bandwidth, and the communication system is said to be operating in the **power-limited regime**. In contrast, for small \( W \), \( \frac{dC}{dW} \) is quite large and the capacity is very sensitive to changes in the bandwidth. Such communication systems are said to be operating in the **bandwidth-limited regime**.

(b) What SNR is required to achieve a capacity of 56 kbits/sec over a channel of bandwidth 3300 Hz?

**Solution:** \( 56000 = 3300 \log_2 \left( 1 + \frac{P}{N_0W} \right) \) for \( \frac{P}{N_0W} \approx 51 \text{ dB} \). The bandwidth is that of a voice-grade telephone-line (wire) channel, and 56 kbps was once the state-of-the-art for modems operating over telephone lines.
(c) The thermal noise power in a system of bandwidth $W$ is $4kTW$ Watts where $k$ is Boltzmann’s constant $(1.38 \times 10^{-23} \text{Joules/K})$ and $T$ is the absolute temperature in $K$. If $T = 300K$, what value of $P$ is required in part (b)?

**Solution:** $N_0W = 4kTW = 5.46 \times 10^{-17}$ so that the minuscule $P \approx 7 \times 10^{-12}$ Watts suffices. The data rates achievable over voice-grade telephone line channels are limited not by the power (more than adequate amounts are always available!) but rather by intersymbol interference. These channels operate in the bandwidth-limited regime.

7. **[On Bandwidth]**
Read David Slepian’s Shannon Lecture, “On Bandwidth”. Then write a short essay (2-3 paragraphs) on the value of mathematical models in engineering design. Your writing will be graded along the following dimensions: (a) Content, (b) Organization, (c) Development, and (d) Use of Language.

**Solution:** We use the following scoring rubric.

<table>
<thead>
<tr>
<th>Rating</th>
<th>Content</th>
</tr>
</thead>
<tbody>
<tr>
<td>Top</td>
<td>Answer is appropriate to the question. Content is factually correct.</td>
</tr>
<tr>
<td>High</td>
<td>Answer is appropriate to the question. Content may have one or two errors.</td>
</tr>
<tr>
<td>Medium</td>
<td>Content relates peripherally to the question; contains significant factual errors.</td>
</tr>
<tr>
<td>Low</td>
<td>Content unrelated to question.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Rating</th>
<th>Organization</th>
</tr>
</thead>
<tbody>
<tr>
<td>Top</td>
<td>Clear sense of order. Supporting points are presented in a logical progression.</td>
</tr>
<tr>
<td>High</td>
<td>Points are presented in a logical progression.</td>
</tr>
<tr>
<td>Medium</td>
<td>Logic of argument is minimally perceivable. Points presented in a seemingly random fashion.</td>
</tr>
<tr>
<td>Low</td>
<td>Lacks clear organizational plan, confusing.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Rating</th>
<th>Development</th>
</tr>
</thead>
<tbody>
<tr>
<td>Top</td>
<td>Develops each point with may specific details. Answers question completely.</td>
</tr>
<tr>
<td>High</td>
<td>Each point supported with some details and evidence. All important points included.</td>
</tr>
<tr>
<td>Medium</td>
<td>Sparse details or evidence. Question only partially answered.</td>
</tr>
<tr>
<td>Low</td>
<td>Statements are unsupported by any explanation. Repetitious, incoherent, illogical.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Rating</th>
<th>Use of Language</th>
</tr>
</thead>
<tbody>
<tr>
<td>Top</td>
<td>Uses technical terminology appropriately and correctly. No grammatical or spelling errors.</td>
</tr>
<tr>
<td>High</td>
<td>Accurate word choice. No more than 2 major errors and a few minor errors.</td>
</tr>
<tr>
<td>Medium</td>
<td>Use of technical terminology avoided. Some serious errors (but don’t impair communication).</td>
</tr>
<tr>
<td>Low</td>
<td>Limited vocabulary; errors impair communication.</td>
</tr>
</tbody>
</table>

8. **[Errors]**
Please write down any errors you have seen in the course notes.