ECE 361: Problem Set 4
Capacity and Sampling

Released: Thursday, March 3
Due: Thursday, March 10 (in class)
Reading: 361 Course Notes Lectures 11-13.

1. **[BEC with Feedback]**

Suppose we are communicating over a BEC and there is feedback, as present in internet communication. In particular, assume the transmitter is told the output of the channel, \(y[1], \ldots, y[i-1]\) at time \(i\), so that it knows which transmitted symbols were erased. Suppose the probability of erasure is \(p\), and we use the following transmission scheme:

- Transmit \(n\) symbols,
- We know approximately \(pn\) were erased, so we retransmit them.
- We know approximately \(p\) fraction of these were erased, so we retransmit them.
- We continue following the third step until all symbols have been received unerased.

(a) In expectation, how many total transmissions are required until all symbols are received without erasure?

(b) What is the (expected) communication rate of this scheme?

2. **[Minimum SNR for Reliable Communication]**

With an average energy constraint of \(E\), the capacity of the discrete-time Gaussian channel was noted to be \(C = \frac{1}{2} \log_2 (1 + \text{SNR})\) where \(\text{SNR} = \frac{E}{\sigma^2}\). Now consider a communication system operating at rate \(R\) very close to \(C\) with very high reliability.

(a) Show that the bit energy \(E_b\) equals \(E/R\).

(b) Show that \(E_b/\sigma^2 \approx 2\text{SNR}/\log_2 (1 + \text{SNR})\) which is an increasing function of \(\text{SNR}\).

(c) Naturally, we want to operate with as small a value of \(E_b\) as possible.

Show that \(\lim_{\text{SNR} \to 0} 2\text{SNR}/\log_2 (1 + \text{SNR}) = 2 \ln 2 \approx 1.4\) and so it is possible to operate very close to capacity with very high reliability provided that \(E_b/\sigma^2 > 2 \ln 2\).

3. **[Raised Cosine Pulse]**

The raised cosine pulse in the time domain is given by:

\[
g_{rc}(t) = \left(\frac{t}{T}\right) \cos\left(\frac{\pi \beta t}{T} \right) \frac{1}{1 - \frac{4\beta^2 t^2}{T^2}}
\]

In this problem you will establish that its Fourier transform is given by

\[
G_{rc}(f) = \begin{cases} 
\frac{T}{2} \left[ 1 + \cos\left(\frac{\pi T}{2} \left( |f| - \frac{1-\beta}{2T} \right) \right) \right] & 0 \leq |f| \leq \frac{1-\beta}{2T} \\
0 & \frac{1-\beta}{2T} \leq |f| \leq \frac{1+\beta}{2T} \\
|f| > \frac{1+\beta}{2T} 
\end{cases}
\]

(a) First show that \(g_{rc}(t)\) can be rewritten as:

\[
g_{rc}(t) = \left(\frac{t}{T}\right) b(t)
\]

where

\[
b(t) = \frac{\pi}{4} \left[ \sin\left(\frac{\beta t}{T} + \frac{1}{2}\right) + \sin\left(\frac{\beta t}{T} - \frac{1}{2}\right) \right]
\]
(b) Next show that the Fourier transform of $b(t)$ is given by

$$B(f) = \frac{\pi T}{2\beta} \cos\left(\frac{\pi T f}{\beta}\right) \text{rect}\left(\frac{T f}{\beta}\right)$$

(c) Now find $G_{rc}(f)$ by convolving $B(f)$ with $T\text{rect}(T f)$.

*Hint:* Only consider $f \geq 0$ in computing the convolution, arguing first that $G_{rc}(f)$ is an even function.

4. **[Minimum Energy per bit and Optimality of Position Modulation]**

Consider communication on a continuous-time AWGN channel with bandwidth $W$ and power $P$. The energy per bit that is consumed if we transmit bits at rate $R$ bits/s is given by

$$E_b = \frac{P}{R}$$

For a fixed bandwidth $W$, the minimum energy per bit is therefore obtained when we communicate at the capacity, i.e., when

$$R = C_{\text{AWGN}} = \frac{W}{2} \log_2 \left(1 + \frac{2P}{N_0 W}\right) \text{ bits/s.}$$

and the corresponding energy per bit is given by

$$E_b(W) = \frac{2P}{W \log_2 \left(1 + \frac{2P}{N_0 W}\right)}$$

(a) Show that the minimum value of $E_b(W)$, minimized over $W$, is given by $E_{b,\text{min}} = N_0 \ln 2$.

(b) Now consider applying the position modulation scheme of Lecture 5 of the notes on the continuous time AWGN channel with bandwidth $W$, by using the pulse shaping and sampling scheme discussed in Lecture 9 of the notes. We know from Lecture 5 that the minimum energy per bit required for reliable position modulation on the discrete-time AWGN channel with noise variance $\sigma^2$ is given by $2\sigma^2 \ln 2$. Now, assuming that the samples are sent at the the Nyquist rate of $W$ samples per second on the continuous-time AWGN channel (using the ideal sinc pulse), argue that position modulation achieves the $E_{b,\text{min}}$ of part (a).

5. **[Prolate Spheroidal Wave Functions]**

This problem considers pulse-shaping.

(a) In Lecture, we played around with the tradeoff between bandwidth and time extent in the choice of raised cosine pulse shapes (see posted matlab code), but did not formalize a definition of bandwidth for time-limited signals. Define a precise notion of bandwidth (also write this down for us), and plot this bandwidth as a function of the parameter $\beta$.

(b) In Lecture, we mentioned the prolate spheroidal wave function as a good pulse shape. The posted matlab code computes a discrete-time approximation. Try to study the bandwidth of this pulse shape in the same way as for the raised cosine. It would probably make sense to use parameter values $W = 0.1$ and less.

6. **[Gaussian Channel Capacity]**

Suppose signals of average power $P$ are used over a white Gaussian noise channel with power spectral density $N_0/2$ Watts/Hz and with bandwidth $W$ Hz. Then the channel capacity $C$ is given by:

$$C = W \log_2 \left(1 + \frac{P}{N_0 W}\right) \text{ bits/second.}$$

(a) For fixed SNR = $P/N_0$, as $W$ increases without bound (tends to $\infty$), does $C$ also increase without bound? If not, what is $\lim_{W \to \infty} C$ and does $C$ approach its limiting value monotonically from below, monotonically from above, or in some non-monotone fashion?
(b) What SNR is required to achieve a capacity of 56 kbits/sec over a channel of bandwidth 3300 Hz?

(c) The thermal noise power in a system of bandwidth $W$ is $4kTW$ Watts where $k$ is Boltzmann’s constant $(1.38 \times 10^{-23}$ Joules/K) and $T$ is the absolute temperature in K. If $T = 300K$, what value of $P$ is required in part (b)?

7. [On Bandwidth]
Read David Slepian’s Shannon Lecture, “On Bandwidth”. Then write a short essay (2-3 paragraphs) on the value of mathematical models in engineering design. Your writing will be graded along the following dimensions: (a) Content, (b) Organization, (c) Development, and (d) Use of Language.

8. [Errors]
Please write down any errors you have seen in the course notes.