ECE 361: Problem Set 2
Decisions and Reliable Communication

1. [Detection through Uniform Noise]
Consider a binary communication system where the received signal \( Y \) is:
\[
Y = X + W,
\]
where \( X \) can take values \( a_1 = +1 \) or \( a_2 = -1 \), and \( W \) is uniformly distributed such that the conditional density functions are
\[
f_{Y|X}(y|x = a_1) = \begin{cases} 
\frac{1}{2} & -0.1 \leq y \leq 1.9 \\
0 & \text{otherwise}
\end{cases}
\]
and
\[
f_{Y|X}(y|x = a_2) = \begin{cases} 
\frac{1}{2} & -1.9 \leq y \leq 0.1 \\
0 & \text{otherwise}
\end{cases}
\]
(a) Find the probability of error \( P_e \) for the case of equally likely signals \( X \in \{a_1, a_2\} \), when using an optimal decision rule.
(b) Find the probability of error \( P_e \) for the case when \( x = a_1 \) has probability \( \frac{3}{4} \) and \( x = a_2 \) has probability \( \frac{1}{4} \), when using an optimal decision rule.
(c) More generally, find the probability of error \( P_e \) for the case when \( x = a_1 \) has probability \( \alpha \) and \( x = a_2 \) has probability \( 1 - \alpha \), when using an optimal decision rule.
(d) Now suppose that we use the optimal decision rule from part (c), i.e. the signals have probabilities \( \alpha \) and \( 1 - \alpha \). But we are in a setting of mismatch, so that the signals are actually sent with probabilities \( \beta \) and \( 1 - \beta \) instead. What is the error probability \( P_e \) for this setting?

2. [Detection through Gaussian Noise]
Consider a binary communication system where the received signal \( Y \) is:
\[
Y = X + W,
\]
where \( X \) can take values \( a_1 = +1 \) or \( a_2 = -1 \), and \( W \) is a zero-mean Gaussian random variable with variance 0.1.
(a) Find the optimal decision rule for the case of equally likely signals \( X \in \{a_1, a_2\} \).
(b) Find the corresponding probability of error \( P_e \), as a number.
(c) Find the optimal decision rule when \( x = a_1 \) has probability \( \frac{3}{4} \) and \( x = a_2 \) has probability \( \frac{1}{4} \).
(d) Find the corresponding probability of error \( P_e \), as a number.

3. [Quantizing a Gaussian Random Variable]
Consider a zero-mean, unit-variance Gaussian random variable.
(a) Suppose we quantize the variable so that if it takes values in \( (-\infty, 0) \) we call it \( a_1 \) and if it takes values in \( [0, \infty) \) we call it \( a_2 \). What are the probabilities that each of the \( \{a_i\} \) occur?
(b) Suppose we quantize the variable so that if it takes values in \((-\infty, -1)\) we call it \(a_1\); if it takes values in \([1, \infty)\) we call it \(a_2\); and if it takes values in \([-1, 1]\) we call it \(a_0\). What are the probabilities that each of the \(\{a_i\}\) occur? Express your answer both in terms of the Q function and numerically.

(c) How much entropy is this quantized source generating, expressed numerically?

(d) Design a binary Huffman code for a length-2 superletter constructed from a pair of such quantized variables, and compute the average codeword length normalized to the number of original letters rather than superletters. Express your result numerically.

4. \textbf{[Discrete-time Poisson Channel]}

The discrete-time Poisson (DTP) channel is commonly used to model low-intensity, direct detection optical communication channels, such as for visible light communication. For this channel model, the intensity in the input signal is allowed to vary between discrete time slots while remaining fixed inside each interval, and the channel output is a statistic on the number of received photons in each time interval. Specifically, a channel input intensity \(x\) [photons/second] is corrupted by the combined impact of background radiation and photodetector dark current at a rate \(\lambda\) [photons/second]. The channel output \(y\) [photons] is a random variable that obeys Poisson statistics:

\[
f_{Y|X}(y|x) = \frac{(x + \lambda)^y}{y!} e^{-(x+\lambda)}, \quad x > 0, y \in \mathbb{Z}^+.
\]

Constraints are typically placed on both the peak energy, i.e. \(A\):

\[0 \leq x \leq A,
\]

and the average optical power, i.e. \(\epsilon\):

\[E[x] \leq \epsilon.
\]

Here let us suppose we signal with one symbol at \(x = 0\) and another at \(x = A\). Further, we use the \(A\)-symbol with probability \(\epsilon/A\) and the 0-symbol with probability \(1 - \epsilon/A\).

(a) What is the average optical power \(E[x]\)?

(b) Derive the MAP decision rule for this channel.

5. \textbf{[Repeated uses of binary symmetric channel.]}

Suppose we have a binary symmetric channel with crossover probability \(\varepsilon\), as we had used in our think-pair-share exercise during Lecture 3. The 0 symbol was transmitted with probability \(p\), and the 1 symbol was transmitted with probability \(1 - p\).

(a) Suppose we observe a 1 at the output of the channel. What is the conditional probability, which we call \(p_1\), that the transmitted symbol was a 1?

(b) Suppose we send the same symbol through the channel again, and again observe a 1 at the output of the channel. Let \(p_2\) be the conditional probability that the transmitted symbol was 1, given these two observations of a 1 at the output. Find a simple expression for \(p_2\) in terms of \(p_1\) (and \(\varepsilon\)).

(c) Repeating, let \(p_n\) be the conditional probability that the transmitted symbol was 1, given \(n\) observations of 1 at the output. Find a simple expression for \(p_n\) in terms of \(p\), \(\varepsilon\), and \(n\).

(d) What is \(\lim_{n \to \infty} p_n\)?

6. \textbf{[Repetition coding with majority logic decoding.]}

Consider repetition coding discussed in Lecture 4 notes. There we showed that the ML rule is based on averaging the received voltages at the different instants. Now consider the following majority logic based receiver. Assume that a single bit is being communicated via repetition coding over \(n\) time
instants. The transmit voltage $x$ at all the $n$ time instants is $+\sqrt{E}$ if the bit is 1, and the transmit voltage at all time instants is $-\sqrt{E}$ if the bit is 0. Define, for each time $m$,

$$\hat{y}[m] = \begin{cases} 1 & \text{if } y[m] > 0 \\ -1 & \text{else} \end{cases}$$

Now decide that the transmitted bit is 1 if

$$\sum_{m=1}^{n} \hat{y}[m] > 0$$

and decide that the transmitted bit is 0 if

$$\sum_{m=1}^{n} \hat{y}[m] < 0$$

and decide randomly (with equal probability) between 0 and 1 if

$$\sum_{m=1}^{n} \hat{y}[m] = 0.$$

You can interpret this rule as making an estimate of transmitted bit at each of the time separately and then taking the majority vote to decide what the transmitted bit was.

(a) Derive a formula for the error probability of this majority logic rule. Your formula should only involve the quantities $p = Q(\sqrt{\text{SNR}})$ and $n$, where $\text{SNR} = \sqrt{E}/\sigma$. (You may want to consider the cases of $n$ odd and $n$ even separately.)

(b) Plot error probability of part (a) as a function of $n$, for $n = 1, \ldots, 10$. Assume that $\text{SNR} = 5$ in your plot. Use a log scale on the vertical axis of your plot.

(c) Based on the plot, comment on the nature of the decay of the error probability with $n$, making a comparison to the error decay for ML decoding.

7. [Errors]

Please write down any errors you have seen in the course notes.