ECE 361: Problem Set –: Problems and Solutions
Extra Problems for Preparing for Exam 2

1. [Matched Filter]
   Consider the 3-tap ISI channel with
   \[ y[m] = x[m] + \frac{1}{2} x[m - 1] + \frac{1}{3} x[m - 2] + w[m] \]
   Only two symbols \( x[1] \) and \( x[2] \) are sent on this channel, so you can assume \( x[m] = 0 \) for \( m \neq 1,2 \).

   Assume that \( x[1] \) and \( x[2] \) are i.i.d. with mean 0 and variance \( E \), and the \( w[m] \) are i.i.d. \( \mathcal{N}(0,\sigma^2) \). As usual we define \( \text{SNR} = \frac{E}{\sigma^2} \).

   Our goal is to detect \( x[1] \) from \( y[1], y[2], y[3] \), using a linear equalizer that combines them in a linear fashion to form
   \[ \hat{y}[1] = \sum_{k=1}^{3} c_k y[k] \]

   (a) Find the coefficients \( c_1, c_2, c_3 \) of the matched filter equalizer
   **Solution:** By definition, \( c_1^{\text{MF}} = 1, c_2^{\text{MF}} = \frac{1}{2} \) and \( c_3^{\text{MF}} = \frac{1}{3} \).

   (b) Find the SINR of the matched filter as a function of \( \text{SNR} \).
   **Solution:** Since:
   \[
   \begin{align*}
   \end{align*}
   \]
   we have:
   \[ \hat{y}_{\text{MF}}[1] = (1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{3}\right)^2) x[1] + \left(\frac{1}{2} + \frac{1}{6}\right) x[2] + w[1] + \frac{1}{2} w[2] + \frac{1}{3} w[3]. \]
   Thus:
   \[ \text{SINR}_{\text{MF}} = \frac{(1 + \frac{1}{4} + \frac{1}{9})^2 E}{(\frac{1}{2} + \frac{1}{4} + \frac{1}{9})^2 E + (1 + \frac{1}{4} + \frac{1}{9}) \sigma^2} = \frac{\frac{49}{72} \text{SNR}}{\frac{49}{72} \text{SNR} + \frac{1}{18} \text{SNR} + 1}. \]

2. [Zero Forcing Equalizer]
   Consider the same setting as in Problem 1. Here the goal is to derive the zero-forcing equalizer for \( x[1] \).

   (a) Find the relationship between \( c_2 \) and \( c_3 \) that any zero-forcing solution must satisfy.
   **Solution:**
   \[ \hat{y}[1] = (c_1 + \frac{1}{2} c_2 + \frac{1}{3} c_3) x[1] + (c_2 + \frac{1}{2} c_3) x[2] + c_1 w[1] + c_2 w[2] + c_3 w[3] \]  \tag{1}
   Any zero forcing solution must satisfy:
   \[ c_2 + \frac{1}{2} c_3 = 0 \quad \Rightarrow \quad c_3 = -2 c_2. \]

   (b) Use the result of part (a) to write down the SINR of a zero-forcing solution in terms of \( c_1 \) and \( c_2 \).
Solution: Replacing $c_3$ with $-2c_2$ in (??) we get:

$$\hat{y}_{ZF}[1] = (c_1 - \frac{1}{6}c_2)x[1] + c_1w[1] + c_2w[2] - 2c_2w[3].$$

Therefore

$$\text{SINR}_{ZF} = \frac{(c_1 - \frac{1}{6}c_2)^2 E}{(c_1^2 + c_2^2 + 4c_2^2)\sigma^2} = \frac{(c_1 - \frac{1}{6}c_2)^2}{(c_1^2 + 5c_2^2)} \text{SNR.}$$

(c) Use the Cauchy-Schwarz inequality to maximize the SINR as a function of $c_1$ and $c_2$, and find the resulting value of the SINR as a function of $\text{SNR}$

Solution: If we take $\vec{b} = [c_1 \sqrt{5c_2}]^\top$ and $\vec{a} = [1 - \frac{1}{6\sqrt{5}}]^\top$, maximizing SINR as a function of $c_1$ and $c_2$ is equivalent to finding $c_1$ and $c_2$ that jointly maximize

$$\frac{(\vec{a}^\top \vec{b})^2}{\vec{b}^\top \vec{b}}.$$

By the Cauchy-Schwarz inequality

$$(\vec{a}^\top \vec{b})^2 \leq (\vec{a}^\top \vec{a})(\vec{b}^\top \vec{b})$$

for all $\vec{a}, \vec{b}$, and equality is obtained if and only if $\vec{b} = \alpha\vec{a}$ for some nonzero constant $\alpha$, which we can set to 1. Thus, the SINR as a function of $c_1$ and $c_2$ is maximized by setting

$$[c_1 \sqrt{5c_2}]^\top = [1 - \frac{1}{6\sqrt{5}}]^\top.$$

By solving this equation we get $c_1 = 1$, $c_2 = -\frac{1}{30}$ (and $c_3 = -2c_2 = \frac{1}{15}$). The maximum value of SINR equals $\vec{a}^\top \vec{a} \text{SNR} = \frac{181}{180} \text{SNR}$.

3. [MMSE Equalizer]

Consider the same setting as in Problem 1. Here the goal is to derive the MMSE equalizer for $x[1]$.

(a) Find the linear transformation of the channel outputs that whitens the interference plus noise terms in the channel outputs.

Solution:


where


Clearly, $z[2]$ and $z[3]$ are correlated since $x[2]$ appears in both of them. Thus, we want to find an appropriate linear combination of $y[2]$ and $y[3], i.e.


such that


is uncorrelated with $z[2], i.e.,

$$0 = \text{cov}(\hat{z}[3], z[2]) = E[\hat{z}[3]z[2]] - E[\hat{z}[3]]E[z[2]] = E[\hat{z}[3]z[2]] = E[z[3]z[2]] - aE[(z[2])^2],$$

where, in the second last equality, we have used the fact that $z[2]$ (and in fact $\hat{z}[3]$ too) is zero mean. Therefore, $a$ must satisfy

$$\frac{1}{2} E = a(E + \sigma^2) \quad \Rightarrow \quad a = \frac{\text{SNR}}{2 \text{SNR} + 1}. \quad (2)$$

\[
\hat{y}[3] = y[3] - \frac{\text{SNR}}{\text{SNR}+1} y[2],
\]

whitens the interference plus noise terms in the channel outputs.

(b) Now scale the modified outputs you found in part (a) to normalize the variances in the interference plus noise terms.

**Solution:** From part (a), we have

\[
y[1] = x[1] + z[1], \quad y[2] = \frac{1}{2} x[1] + z[2], \quad \hat{y}[3] = (\frac{1}{3} - \frac{1}{2} a) x[1] + \hat{z}[3],
\]

where \( z[1], z[2] \) and \( \hat{z}[3] \) are uncorrelated, and \( a \) is given in (??). We now let \( a_2 \) and \( a_3 \) be the normalizing factors applied to \( y[2] \) and \( \hat{y}[3] \), respectively, so that the variances of the noise parts of \( \hat{y}[2] = a_2 y[2] \) and \( \hat{y}[3] = a_3 \hat{y}[3] \) will both equal that of \( \hat{y}[1] = y[1] \), which is \( \sigma^2 \). Thus

\[
a_2^2 \sigma_{z[2]}^2 = a_3^2 \sigma_{\hat{z}[3]}^2 = \sigma^2.
\]

Now

\[
\sigma_{z[2]}^2 = E + \sigma^2
\]

and

\[
\]

where the first equality comes from \( \hat{z}[3] \) being zero mean. Moreover, since \( x[2] \) and \( w[3] - aw[2] \) are both zero mean and uncorrelated, we can further simplify the above equation to get

\[
\sigma_{\hat{z}[3]}^2 = (\frac{1}{2} - a)^2 E + \sigma^2 (1 + a^2).
\]

Thus

\[
a_2^2 = \frac{\sigma^2}{\sigma^2 + E} = \frac{1}{1 + \text{SNR}} \quad \Rightarrow \quad a_2 = \sqrt{\frac{1}{1 + \text{SNR}}},
\]

and after replacing \( a \) with \( \frac{\text{SNR}}{\text{SNR}+1} \), we finally get

\[
a_3 = \frac{2(\text{SNR} + 1)}{\sqrt{5\text{SNR}^2 + 9\text{SNR} + 4}}.
\]

To summarize:

\[
\hat{y}[1] = y[1] \quad \hat{y}[2] = \sqrt{\frac{1}{1 + \text{SNR}}} y[2], \quad \hat{y}[3] = \frac{2(\text{SNR} + 1)}{\sqrt{5\text{SNR}^2 + 9\text{SNR} + 4}} \hat{y}[3].
\]

(c) Now find the linear combination of the scaled outputs you found in (b) that is the MMSE equalizer.

Note that the coefficients of this linear combination can be a function of SNR.

**Solution:** The MMSE solution, as a linear combination of the scaled output found in (b), is thus:

\[
\hat{y}_{\text{MMSE}}[1] = \hat{y}[1] + \frac{1}{2} a_2 \hat{y}[2] + a_3 (\frac{1}{3} - \frac{1}{2} a) \hat{y}[3],
\]

where \( a_2, a_3 \) and \( a \) have all been given explicitly above.

4. [Matched Filter]

Consider the 3-tap ISI channel with

\[
y[m] = x[m] + \frac{1}{3} x[m - 1] + \frac{1}{4} x[m - 2] + w[m]
\]
5. [Zero Forcing Equalizer]

Consider the same setting as in Problem 4. Here the goal is to derive the zero-forcing equalizer for $x[1]$. 

(a) Find the coefficients $c_1, c_2, c_3$ of the matched filter equalizer.

**Solution:** By definition, $c_1^{MF} = 1$, $c_2^{MF} = \frac{1}{2}$, and $c_3^{MF} = \frac{1}{4}$.

(b) Find the SINR of the matched filter as a function of $SNR$.

**Solution:** Since:

$$\hat{y}_{MF}[1] = (1 + \frac{1}{2} + \frac{1}{10})^2 E + (1 + \frac{1}{9} + \frac{1}{10})^2 \sigma^2 = (\frac{169}{107})^2 SNR$$

we have:

$$\text{SINR}_{MF} = \frac{(1 + \frac{1}{9} + \frac{1}{10})^2 E}{(\frac{169}{107})^2} = \frac{(\frac{169}{107})^2 SNR}{\frac{169}{107} SNR + 1}$$

5. [Zero Forcing Equalizer]

Consider the same setting as in Problem 4. Here the goal is to derive the zero-forcing equalizer for $x[1]$.

(a) Find the relationship between $c_2$ and $c_3$ that any zero-forcing solution must satisfy.

**Solution:**

$$\hat{y}[1] = (c_1 + \frac{1}{3} c_2 + \frac{1}{4} c_3) x[1] + (c_2 + \frac{1}{3} c_3) x[2] + c_1 w[1] + c_2 w[2] + c_3 w[3]$$  \hspace{1cm} (3)

Any zero foring solution must satisfy:

$$c_2 + \frac{1}{3} c_3 = 0 \implies c_3 = -3c_2.$$  

(b) Use the result of part (a) to write down the SINR of a zero-forcing solution in terms of $c_1$ and $c_2$.

**Solution:** Replacing $c_3$ with $-3c_2$ in (3) we get:

$$\hat{y}_{ZF}[1] = (c_1 - \frac{5}{12} c_2) x[1] + c_1 w[1] + c_2 w[2] - 3c_2 w[3].$$

Therefore

$$\text{SINR}_{ZF} = \frac{(c_1 - \frac{5}{12} c_2)^2 E}{(c_1^2 + c_2^2 + 9c_2^2) \sigma^2} = \frac{(c_1 - \frac{5}{12} c_2)^2}{(c_1^2 + 10c_2^2)} \text{SNR}.$$  

(c) Use the Cauchy-Schwarz inequality to maximize the SINR as a function of $c_1$ and $c_2$, and find the resulting value of the SINR as a function of $SNR$.

**Solution:** If we take $\vec{b} = [c_1 \sqrt{10} c_2] \top$ and $\vec{a} = [1 \frac{5}{12\sqrt{10}}] \top$, maximizing SINR as a function of $c_1$ and $c_2$ is equivalent to finding $c_1$ and $c_2$ that jointly maximize

$$\frac{(\vec{a} \top \vec{b})^2}{\vec{b} \top \vec{b}}.$$
By the Cauchy-Schwarz inequality
\[(\vec{a}^\top \vec{b})^2 \leq (\vec{a}^\top \vec{a})(\vec{b}^\top \vec{b})\]
for all \(\vec{a}, \vec{b}\), and equality is obtained if and only if \(\vec{b} = \alpha \vec{a}\) for some nonzero constant \(\alpha\), which we can set to 1. Thus, the SINR as a function of \(c_1\) and \(c_2\) is maximized by setting

\[c_1 \sqrt{10}c_2 = [1 - \frac{5}{12\sqrt{10}}]^\top.\]

By solving this equation we get \(c_1 = 1\), \(c_2 = -\frac{1}{24}\) (and \(c_3 = -3c_2 = \frac{1}{8}\)). The maximum value of SINR equals \(\vec{a}^\top \vec{a} \text{SNR} = \frac{293}{288}\).

6. **[MMSE Equalizer]**

Consider the same setting as in Problem 4. Here the goal is to derive the MMSE equalizer for \(x[1]\).

(a) Find the linear transformation of the channel outputs that whitens the interference plus noise terms in the channel outputs.

**Solution:**


Clearly, \(z[2]\) and \(z[3]\) are correlated since \(x[2]\) appears in both of them. Thus, we want to find an appropriate linear combination of \(y[2]\) and \(y[3]\), i.e.


such that


is uncorrelated with \(z[2]\), i.e.,

\[0 = \text{cov}(\hat{z}[3], z[2]) = E[\hat{z}[3]z[2]] - E[\hat{z}[3]]E[z[2]] = E[z[3]z[2]] - E[z[3]]E[z[2]] - aE[(z[2])^2],\]

where, in the second to last equality, we have used the fact \(z[2]\) (and in fact \(\hat{z}[3]\) too) is zero mean. Therefore, \(a\) must satisfy

\[\frac{1}{3}E = a(E + \sigma^2) \implies a = \frac{\frac{1}{3}\text{SNR}}{\text{SNR} + 1}.\]

The linear combination of \(y[3]\) and \(y[2]\):

\[\hat{y}[3] = y[3] - \frac{\frac{1}{3}\text{SNR}}{\text{SNR} + 1}y[2],\]

whitens the interference plus noise terms in the channel outputs.

(b) Now scale the modified outputs you found in part (a) to normalize the variances in the interference plus noise terms.

**Solution:** From part (a), we have

\[y[1] = x[1] + z[1], y[2] = \frac{1}{3}x[1] + z[2], \hat{y}[3] = (\frac{1}{4} - \frac{1}{3}a)x[1] + \hat{z}[3],\]

where \(z[1], z[2]\) and \(\hat{z}[3]\) are uncorrelated, and \(a\) is given in (4). We now let \(a_2\) and \(a_3\) be the normalizing factors applied to \(y[2]\) and \(\hat{y}[3]\), respectively, so that the variances of the noise parts of \(\hat{y}[2] = a_2y[2]\) and \(\hat{y}[3] = a_3\hat{y}[3]\) will both equal that of \(\hat{y}[1] = y[1]\), which is \(\sigma^2\). Thus

\[a_2^2\sigma^2_{\hat{y}[2]} = a_3^2\sigma^2_{\hat{y}[3]} = \sigma^2.\]
Now \( \sigma^2_{z[2]} = E + \sigma^2 \)

and

\[
\sigma^2_{z[3]} = \mathbb{E}[\hat{z}[3]^2] = \mathbb{E}[\hat{z}[3] - a\hat{z}[2]^2] = \mathbb{E}[(\frac{1}{3} - a)x[2] + (w[3] - aw[2])^2],
\]

where the first equality comes from \( \hat{z}[3] \) being zero mean. Moreover, since \( x[2] \) and \( w[3] - aw[2] \) are both zero mean and uncorrelated, we can further simplify the above equation to get

\[
\sigma^2_{z[3]} = (\frac{1}{3} - a)^2E + \sigma^2(1 + a^2).
\]

Thus

\[ a_2^2 = \frac{\sigma^2}{\sigma^2 + E} = \frac{1}{1 + \text{SNR}} \quad \Rightarrow \quad a_2 = \sqrt{\frac{1}{1 + \text{SNR}}}, \]

and after replacing \( a \) with \( \frac{\text{SNR}}{\text{SNR} + 1} \), we finally get

\[ a_3^2 = \frac{\text{SNR} + 1}{(1 + \text{SNR})^2 + \frac{1}{3}\text{SNR}}. \]

and so

\[ a_3 = \sqrt{\frac{\text{SNR} + 1}{(1 + \text{SNR})^2 + \frac{1}{3}\text{SNR}}}. \]

To summarize:

\[
\hat{y}[1] = y[1] \quad \hat{y}[2] = \sqrt{\frac{1}{1 + \text{SNR}}} y[2], \quad \hat{y}[3] = \sqrt{\frac{\text{SNR} + 1}{(1 + \text{SNR})^2 + \frac{1}{3}\text{SNR}}} \hat{y}[3].
\]

(c) Now find the linear combination of the scaled outputs you found in (b) that is the MMSE equalizer. Note that the coefficients of this linear combination can be a function of \( \text{SNR} \).

**Solution:** The MMSE solution, as a linear combination of the scaled output found in (b), is thus:

\[
\hat{y}_{\text{MMSE}}[1] = \hat{y}[1] + \frac{1}{3}a_2\hat{y}[2] + a_3(\frac{1}{4} - \frac{1}{3}a)\hat{y}[3],
\]

where \( a_2, a_3 \) and \( a \) have all been given explicitly above.

7. **[Intersymbol Interference Again]**

Consider the 2-tap ISI channel with

\[
y[m] = h_0x[m] + h_1x[m-1] + w[m]
\]

where the \( \{w[m]\} \) are independent Gaussian noise variables with variance \( \sigma^2 \). Further suppose \( x[0] = \pm \sqrt{E} \) with equal probability. With transmission starting at time 0, we observe \( y[0] \) and \( y[1] \).

(a) Suppose we do not transmit anything during time 1, so \( x[1] = 0 \). Give the matched filter receiver to detect \( x[0] \). Give an exact expression for the average probability of detection error in terms of the \( Q \) function.

**Solution:** \( y[0] \) and \( y[1] \) are given as follows:

\[
y[0] = h_0x[0] + w[0] \\
y[1] = h_1x[0] + w[1]
\]

\( \hat{y}[0] \) under the matched filter is, by definition, simply

\[
\hat{y}[0] = h_0y[0] + h_1y[1]
\]
Substituting the values for \(y[0]\) and \(y[1]\) yields:

\[
\hat{y}[0] = h_0y[0] + h_1y[1] \\
= h_0(h_0x[0] + w[0]) + h_1(h_1x[0] + w[0]) \\
= (h_0^2 + h_1^2)x[0] + h_0w[0] + h_1w[1] \\
\]

We declare a 1 as the transmitted signal when \(\hat{y} > 0\), so in this case the probability of error would be:

\[
P_e = Pr[\hat{y} < 0|x[0] = \sqrt{E}] \\
\]

Notice that when \(x[0] = \sqrt{E}\), the distribution for \(\hat{y}[0]\) is \(N((h_0^2 + h_1^2)\sqrt{E}, (h_0^2 + H_1^2)\sigma^2)\), so

\[
P_e = Q \left( \frac{(h_0^2 + h_1^2)\sqrt{E}}{\sqrt{h_0^2 + H_1^2}\sigma} \right) \\
\]

(b) Suppose instead that during time 1, we send another symbol \(x[1] = \pm \sqrt{E}\) with equal probability, independent of \(x[0]\). We use the same detector as in part (a). Given an exact expression for the average probability of detection error in terms of the \(Q\) function. Notice the interference from \(x[1]\) is not Gaussian.

**Solution:** \(y[0]\) and \(y[1]\) are given as follows:

\[
y[0] = h_0x[0] + w[0] \\
y[1] = h_0x[1] + h_1x[0] + w[1] \\
\]

Now, substituting corresponding values for \(y[0]\) and \(y[1]\) yields:

\[
\hat{y}[0] = h_0y[0] + h_1y[1] \\
= h_0(h_0x[0] + w[0]) + h_1(h_0x[1] + h_1x[0] + w[1]) \\
= (h_0^2 + h_1^2)x[0] + (h_1h_0x[1] + h_0w[0] + h_1w[1]) \\
\]

As before, we declare 1 when \(\hat{y} > 0\). Since 0 and 1 are transmitted equiprobably, by symmetry, we have:

\[
P_e = \frac{1}{2} Pr[\hat{y} < 0|x[0] = \sqrt{E}, x[1] = \sqrt{E}] + \frac{1}{2} Pr[\hat{y} < 0|x[0] = \sqrt{E}, x[1] = -\sqrt{E}] \\
\]

Here, when \(x[0] = \sqrt{E}\) and \(x[1] = \sqrt{E}\), the distribution for \(\hat{y}[0]\) is according to \(N((h_0^2 + h_1^2 + h_0h_1)\sqrt{E}, (h_0^2 + h_1^2)\sigma^2)\) and when \(x[0] = \sqrt{E}\) and \(x[1] = -\sqrt{E}\), the distribution for \(\hat{y}[0]\) is according to \(N((h_0^2 + h_1^2 - h_0h_1)\sqrt{E}, (h_0^2 + h_1^2)\sigma^2)\). Therefore:

\[
P_e = \frac{1}{2} Q \left( \frac{(h_0^2 + h_1^2 + h_0h_1)\sqrt{E}}{\sqrt{h_0^2 + h_1^2}\sigma} \right) + \frac{1}{2} Q \left( \frac{(h_0^2 + h_1^2 - h_0h_1)\sqrt{E}}{\sqrt{h_0^2 + h_1^2}\sigma} \right) \\
\]

(c) Continuing with part (b), what happens to performance of the receiver at high SNR?

**Solution:** As we can see in the final expression in the previous part, the probability of error goes to zero for large SNR.

**8. [Naive Precoding]**

Consider the 2-tap ISI channel:

\[
y[m] = x[m] + h_1x[m - 1] + w[m], \quad m \geq 1. \\
\]

Suppose the transmitter does “naive” precoding to send one bit at a time:

\[
x[m] = d[m] - h_1x[m - 1], \quad m \geq 1, \\
\]

where \(d[m]\) is \(\pm \sqrt{E}\) based on whether the information bit at time \(m\) is 0 or 1. Time begins at 1 and you can suppose that \(x[0] = 0\). Also suppose that \(h_1\) is a strictly positive number.
(a) Demonstrate a sequence of \( \{d[m]\} \) for which the energy in the transmit symbol \( x[m] \) increases monotonically as time \( m \) grows.

**Solution:** Consider the sequence \( \{d[m]\} \) where \( d[m] = \sqrt{E} \) if \( m \) is odd and \( -\sqrt{E} \) if \( m \) is even. Then,

\[
x[m] = \pm \left( 1 + h_1 + h_1^2 + \cdots + h_1^{m-1} \right) \sqrt{E}
\]

being positive or negative depending on whether \( m \) is odd or even, respectively. The energy in the transmit symbol \( x[m] \) is

\[
\begin{cases}
\left( \frac{1-h_1^m}{1-h_1} \right)^2 E & \text{if } h_1 \neq 1 \\
M^2 E & \text{if } h_1 = 1.
\end{cases}
\]

(b) What is the largest limiting value of the transmit energy?.

**Solution:** The largest transmit energy (as time grows) is

\[
\frac{E}{(1-h_1)^2} \quad \text{if } 0 < h_1 < 1,
\]

\[
\infty \quad \text{else.}
\]

(c) How does this compare with the average calculation in the lecture notes?

**Solution:** We can compare this to the calculation in the lecture that was based on “averages”, and we see that:

- for \( h_1 \geq 1 \), the largest transmit energy goes to infinity in both cases.
- for \( 0 < h_1 < 1 \) the difference is that the average calculation in the lecture “underestimates” the largest transmit energy by a factor of

\[
\frac{(1-h_1)^2}{1-h_1^2} < 1.
\]

9. **[OFDM Power Allocation]**

Consider the OFDM parallel AWGN channel with \( N = 2 \), and

\[
\hat{y}[0] = \frac{3}{2} x[0] + \hat{w}[0]
\]

\[
\hat{y}[1] = \frac{1}{2} x[1] + \hat{w}[1]
\]

where the noises \( \hat{w}[0] \) and \( \hat{w}[1] \) are i.i.d. \( N(0,1) \) random variables, i.e., \( \sigma^2 = 1 \).

(a) Given a total power constraint \( P \) to use over the channel, derive explicitly the expression for the optimal way to split the power over the two OFDM sub-channels \( (P_0^*, P_1^*) \). Note that the nature of the power allocation will depend on the value of \( P \), and so you need to consider different regions of \( P \) separately in writing down the solution.

**Solution:** Since \( \hat{h}_0 = \frac{3}{4} \) and \( \hat{h}_1 = \frac{1}{4} \), we can use equation (438) in lecture 17 course notes to get:

\[
P_0^* = \left( \frac{1}{\lambda} - \frac{1}{2} \right)^+ \quad \text{and} \quad P_1^* = \left( \frac{1}{\lambda} - \frac{1}{4} \right)^+.
\]

Moreover, \( P_0^* + P_1^* = P \).

Case(i): Suppose

\[
P_1^* = \frac{1}{\lambda} - \frac{1}{4}
\]
Since
\[ \frac{1}{\lambda} - \frac{1}{4} \leq \frac{1}{\lambda} - \frac{7}{4}, \]
we have that
\[ P_0^* = \frac{1}{\lambda} - \frac{1}{4}. \]

Therefore, we can exploit the fact that \( P_0^* + P_1^* = P \) to get:
\[ P_0^* = \frac{P}{2} + \frac{16}{9}, \quad \text{and} \quad P_1^* = \frac{P}{2} - \frac{16}{9}. \]

Note that for the above solution to be valid we must have \( P_1^* > 0 \), which means that \( P \geq \frac{32}{9} \).

To summarize, for \( P \geq \frac{32}{9} \), \( P_0^* = \frac{P}{2} + \frac{16}{9} \) and \( P_1^* = \frac{P}{2} - \frac{16}{9} \).

Case(ii): If \( 0 < P < \frac{32}{9} \), we can see from the above analysis that \( P_1^* = 0 \) and thus \( P_0^* = P \)

(b) What happens to the optimal power allocation when \( P \ll 1 \)?

Solution: When \( P \ll 1 \), case (ii) applies and the best way to allocate the power is to invest all the power in subchannel 0.

(c) What happens to the optimal power allocation when \( P \gg 1 \)?

Solution: When \( P \gg 1 \), the optimal way to allocate the power follows case (i) for large enough \( P \). As \( P \) grows larger and larger, the optimal allocation strategy converges to an equal allocation of power across the two channels.