**The Complementary Unit Gaussian Distribution Function $Q(x)$**

Let $\phi(u)$, where $\phi(u) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right)$, denote the probability density function (pdf) of a standard (or unit) Gaussian random variable. The cumulative probability distribution function (CDF) of this random variable is denoted $\Phi(x)$ where

$$\Phi(x) = \int_{-\infty}^{x} \phi(u)\,du = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) du.$$

$\Phi(x)$ is also known as the *unit Gaussian distribution function* and

$$Q(x) = 1 - \Phi(x) = \int_{x}^{\infty} \phi(u)\,du = \int_{x}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) du$$

is called the *complementary unit Gaussian distribution function.* Note that since $\Phi(x)$ is a monotone *increasing* function rising from 0 at $-\infty$ to 1 at $\infty$, $Q(x)$ is a monotone *decreasing* function falling from 1 at $-\infty$ to 0 at $\infty$. In many applications in communications and signal processing, $Q(x)$ is slightly more convenient to use than $\Phi(x)$. For example, the bit error rate in some communication systems can be expressed as $Q(\sqrt{\text{SNR}})$ where SNR is the *signal-to-noise ratio.* Since $Q(x)$ is a decreasing function of its argument, maximizing the SNR is an important objective in communication system design.

Since the pdf $\phi(u)$ is an even function of $u$, we readily see that $\Phi(0) = Q(0) = \frac{1}{2}$. Furthermore, $\Phi(x)$ and $Q(x)$ enjoy the property:

$$\Phi(-x) = 1 - \Phi(x) \text{ and } Q(-x) = 1 - Q(x) \text{ for all real numbers } x.$$

There is no finite-length formula in terms of more commonly used functions (exp, ln, sin, cos etc.) for the antiderivative of $\phi(u)$, though several infinite series are known. Two such series for $\Phi(x)$ are shown below.

$$\Phi(x) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n! 2^n (2n+1)} = \frac{1}{2} + \phi(x) \sum_{n=0}^{\infty} \frac{x^{2n+1}}{1 \cdot 3 \cdot 5 \cdots (2n+1)} \text{ for } x \geq 0.$$

For any given value of $x$, cumbersome numerical calculations are required to compute even an approximate value for $\Phi(x)$ and $Q(x)$. Therefore, extensive tables of values for these functions have been published, though for many applications, the short table in the text (page 222) suffices. An alternative, used in many scientific calculators, evaluates $Q(x)$ for $x \geq 0$ via a rational function approximation:

$$Q(x) \approx \phi(x)(b_1 t + b_2 t^2 + b_3 t^3 + b_4 t^4 + b_5 t^5) \text{ where } t = \frac{1}{1 + 0.2316419x},$$

$b_1 = 0.319381530$, $b_2 = 0.356563782$, $b_3 = 1.781477937$, $b_4 = 1.821255978$, and $b_5 = 1.330274429$. The magnitude of the error in the approximation is smaller than $7.5 \times 10^{-8}$ for all $x \geq 0$. Of course, $Q(x) < 10^{-9}$ for $x \geq 6$, and thus care should be exercised in evaluating $Q(x)$ via calculators for large values of $x$: the value returned may be meaningless.

**Bounds on $Q(x)$**

For large values of $x$, the value of $\Phi(x)$ is very close to 1, and tabulation takes large amounts of space since most values begin $0.999\cdots$. For this reason, $Q(x)$ is generally tabulated (in scientific notation such as $7.5 \times 10^{-8}$) for large values of $x$. Many communication systems are required to operate at bit error rates that are orders of magnitude smaller than this, and it is of some interest to obtain tight upper and lower bounds on the value of $Q(x)$. We begin by noting that $\frac{d}{du}\phi(u) = -u\phi(u)$, and hence, integrating by parts, we have that for $x > 0$,

$$\begin{aligned} Q(x) &= \int_x^\infty \phi(u)\,du = \int_x^\infty \frac{-1}{u}\cdot(-u\phi(u))\,du = \frac{-1}{u}\phi(u)\Big|_x^\infty - \int_x^\infty \frac{1}{u^2}\cdot\phi(u)\,du \\ &= \frac{1}{x}\phi(x) - \int_x^\infty \frac{1}{u^2}\cdot\phi(u)\,du. \end{aligned}$$

Since the integrand of the integral in the line above is positive, so is the integral. Even though we do not know the exact value of the integral, we can nonetheless deduce that $Q(x) < x^{-1}\phi(x)$ for $x > 0$. Next, writing the integrand above as $\frac{-1}{u^3}\cdot(-u\phi(u))$ and repeating the integration by parts and the argument about the value of an integral being positive, we get that $Q(x) > (x^{-1} - x^{-3})\,\phi(x)$ for $x > 0$. Combining these results, we have that

$$\left(\frac{1}{x} - \frac{1}{x^3}\right)\phi(x) < Q(x) < \frac{1}{x}\phi(x) \quad \text{for } x > 0.$$

These upper and lower bounds are *very tight* for large values of $x$ but approach $\pm\infty$ as $x$ approaches 0, and thus become useless. For small values of $x$, another upper bound is more useful. If $u > x \geq 0$, then $u+x \geq u-x > 0$, and hence $(u+x)(u-x) = u^2-x^2 \geq (u-x)^2 > 0$, with equality holding only when $x = 0$. Hence,

$$\exp(x^2/2)Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{u^2-x^2}{2}\right)du \leq \int_x^\infty \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(u-x)^2}{2}\right)du = \frac{1}{2}$$

where the last step follows from recognizing that the integral is the area to the right of $x$ under the pdf of a unit-variance Gaussian random variable with mean $x$. It follows that

$$Q(x) \leq \frac{1}{2}\exp\left(-\frac{x^2}{2}\right) \quad \text{for } x \geq 0$$

with equality holding only at $x = 0$. For large values of $x$, this is a weaker upper bound on $Q(x)$ than $x^{-1}\phi(x)$, but the simpler bound is invaluable for small values of $x$.

These bounds show that $Q(x)$ is an *exponentially decreasing* function of (the square of) its argument. The rapid decrease in the value of $Q(x)$ can be appreciated by noting that $Q(0) = \frac{1}{2}$, $Q(1) = 0.1587\ldots$, $Q(1.96) = 0.0250\ldots$, $Q(2) = 0.0228\ldots$, $Q(3) = 0.0013\ldots$, and $Q(6) = 0.9866\ldots \times 10^{-9}$. Thus, there is a less than 5% chance that a Gaussian random variable deviates from its mean by 2 or more standard deviations, and "six-sigma" quality means a failure rate of less than two in a billion.