ECE 361: Final Exam
Friday, 9 May 2014, 8:00 am – 11:00 am

Name: (in BLOCK CAPITALS) ________________________________

Signature: ________________________________

Instructions
This is a closed-book closed-notes examination except that one side of one 8.5” × 11” sheet of notes is permitted. Tables of integrals, calculators, computers, wireless telegraphys, wireless telephones, microscopes, etc. are neither needed nor permitted.

SHOW YOUR WORK. If you need additional space, use the back of the previous page. Write your final answers in the spaces provided.

This exam contains six problems on ten separate pages

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1. Consider one-shot binary communication to a receiver with $K$ antennas, where the transmitted signal can take values $\pm \sqrt{E}$. Each antenna demodulates separately and so the received signal at antenna $k$ is

$$v_k = g_k x + w_k,$$

where $x \in \pm \sqrt{E}$, $g_k$ is the gain of antenna $k$, and $w_k \sim \mathcal{N}(0, \sigma^2)$ is noise at antenna $k$. Everything is real-valued and $x, w_1, \ldots, w_K$ are statistically independent. In vector notation, this can be denoted as $\vec{v} = \vec{g}x + \vec{w}$.

(a) Suppose the signal at each antenna $k$ is weighted by an arbitrary real number $q_k$, and then the signals are combined as $y = \sum_k v_k q_k = \langle \vec{v}, \vec{q} \rangle$. What is the maximum likelihood detector for $x$ given $y$?

(b) What is the probability of error for this detector?
(c) Let \( \beta = \langle \vec{g}, \vec{q} \rangle / ||\vec{g}|| ||\vec{q}|| \). Express the probability of error in a form where \( \vec{q} \) does not appear except through its effect on \( \beta \).

(d) Find a \( \vec{q} \) that minimizes the probability of error, over all choices of \( \vec{q} \).
2. Consider the following two functions:

\[
\phi_1(t) = \text{sinc}(t) \cos 2\pi f_c t \\
\phi_2(t) = \text{sinc}(t) \sin 2\pi f_c t
\]

where \( \text{sinc}(t) = \frac{\sin(\pi t)}{\pi t} \).

(a) Sketch the Fourier transform of \( \phi_1(t) \) and label all axes, assuming \( f_c > 1/2 \).

(b) Sketch the Fourier transform of \( \phi_2(t) \) and label all axes, assuming \( f_c > 1/2 \).

(c) What is the inner product of \( \phi_1(t) \) and \( \phi_2(t) \), \( \int \phi_1(t)\phi_2(t)dt \)?
3. Consider the standard two-tap ISI wireline channel with $h_0 = 1$, and $h_1 = 2$ and with $N = 4$. We plan to use OFDM to communicate over this channel. Each of the symbols $\hat{x}[0], \hat{x}[1], \hat{x}[2], \hat{x}[3]$ carry one bit of information with

$$\hat{x}[0] \in \{-1, +1\}, \hat{x}[1] \in \{-1, +1\}, \hat{x}[2] \in \{-2, +2\}, \text{ and } \hat{x}[3] \in \{-2, +2\}.$$

Assume that the channel noises $w[m]$ are i.i.d. $\mathcal{N}(0, 1)$ random variables.

(a) Express the IDFT inputs $\tilde{d}_0, \tilde{d}_1, \tilde{d}_2, \tilde{d}_3$ in terms of $\hat{x}[0], \hat{x}[1], \hat{x}[2], \hat{x}[3]$, ensuring that the IDFT outputs are real-valued.

(b) Find the OFDM channel coefficients $\tilde{h}_0, \tilde{h}_1, \tilde{h}_2, \tilde{h}_3$.

(c) Find the pdf of the induced noise distribution $\tilde{w}_0$. 
(d) Find the probability of error for maximum likelihood decision making on the transmitted bit based on $\tilde{y}_0$. 
4. Consider two standard discrete-time AWGN channels in parallel with the average power constraints $P_1$ and $P_2$, as well as noise variances $\sigma_1^2$ and $\sigma_2^2$.

(a) What is the total Shannon capacity of the parallel channels?

(b) Suppose the average powers used over the two subchannels are fixed at $P_1$ and $P_2$, but that it is possible to cryogenically cool the receiver so as to reduce the noise variances $\sigma_1^2$ and $\sigma_2^2$, but only so much that $\sigma_1^2 > 0$, $\sigma_2^2 > 0$, and

$$\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} \leq \frac{1}{\Sigma},$$

for a specified constant $\Sigma$. In order to maximize the capacity, how should the “coolness” be distributed, expressed as the optimal $\sigma_1^2$ and $\sigma_2^2$?

(c) What is the optimal “coolness” distribution in the very hot regime of $\Sigma \to \infty$, expressed as the optimal $\sigma_1^2$ and $\sigma_2^2$?
(d) What is the optimal “coolness” distribution in the very cold regime of $\Sigma \to 0$, expressed as the optimal $\sigma_1^2$ and $\sigma_2^2$?

(e) Thermal noise variance is usually modeled as a linear function of temperature, say $\sigma^2 = kT$. Suppose the capacity of a discrete-time AWGN channel with average power constraint $P$ and noise variance $\sigma_2$ is measured in nats (using the natural logarithm) rather than bits. How much would an infinitesimal increase in temperature, starting from 300 Kelvin change the channel capacity, expressed in units of nats/Kelvin?
5. Consider a wireless system with a two-antenna transmitter and a one-antenna receiver in flat fading. The equivalent discrete-time baseband model has the form:

\[ y[n] = g_1 x_1[n] + g_2 x_2[n] + z[n] \]

where \( x_i[n] \) is the input to antenna \( j \) at time \( n \), and \( y[n] \) and \( z[n] \) are the channel output and the noise at time \( n \), respectively. The noise is i.i.d. complex-valued with independent \( \mathcal{N}(0, \sigma^2/2) \) real and imaginary parts, for each \( n \). The gains \( g_1 \) and \( g_2 \) are known to the receiver but not the transmitter.

We send a pair of symbols \( u_1 \) and \( u_2 \) over two time slots, mapping into 4 channel uses \( x_i[n] \) for \( i = 1, 2 \) and \( n = 1, 2 \).

If we send symbols in a particular manner corresponding to the Alamouti space-time code, and preprocess \( y[1] \) and \( y[2] \) in a particular manner to get \( \tilde{y}_1 \) and \( \tilde{y}_2 \), the overall relationship we end up with, letting \( ||\tilde{g}||^2 = |g_1|^2 + |g_2|^2 \), is:

\[ \tilde{y}_i = ||\tilde{g}||^2 u_i + w_i, i = 1, 2 \]

where \( w_1 \) and \( w_2 \) have the same joint distribution as \( z[1] \) and \( z[2] \) and both are independent of \( u_1 \) and \( u_2 \).

(a) When \( u_1 \) and \( u_2 \) are drawn independently from the 4-ary constellation \( \{\pm \sqrt{E} \pm j \sqrt{E}\} \), the error probability of an optimum detector for either \( u_i \) is \( P_e = Q(\alpha) \). Determine \( \alpha \) as a function of \( ||\tilde{g}||, E, \sigma^2 \).
(b) Suppose the $g_i$ are independent complex Gaussian random variables with independent $\mathcal{N}(0, 1/2)$ real and imaginary parts, so $||g||^2$ is the sum of two independent exponential random variables and has density

$$f_{||g||^2}(t) = \begin{cases} \frac{t}{e^{-\frac{t}{2}}}, & t > 0 \\ 0, & \text{else.} \end{cases}$$

Show the probability of error averaged over this distribution satisfies

$$\bar{P}_e \leq \beta \left( \frac{E}{\sigma^2} \right)^{-2}$$

where $\beta$ is a constant that does not depend on either $E$ or $\sigma^2$. (Hint: you may use the bound $Q(\alpha) \leq \frac{1}{2} e^{-\frac{\alpha^2}{2}}$.)
6. For each of the following statements, mark whether they are true or false. Justify your answer briefly.

(a) Receive antenna diversity with $L$ antennas results in both a power gain and a rate gain by a factor of $L$ relative to time diversity of order $L$.

TRUE or FALSE?

(b) Consider binary communication on a two-tap ISI channel with $h_0 = 1$. The Tomlinson-Harashima precoding strategy can be applied only if $|h_1| < 1$.

TRUE or FALSE?

(c) In a discrete-time Gaussian channel with fixed noise power (variance) $\sigma^2$ and $\text{SNR} = E/\sigma^2$, doubling the SNR at the receiver will approximately double the channel capacity (measured in bits per channel use) at all SNR.

TRUE or FALSE?

(d) Consider our standard intersymbol interference channel model with $L$ taps. If an overall information packet is broken into $L/2$ subpackets, these subpackets are coded and modulated, and the voltages corresponding to each subpacket are interleaved at the transmitter, then the probability distribution of the interference will be Gaussian.

TRUE or FALSE?